Auctions with Endogenous Initiation*

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Abstract

We study strategic initiation of first-price auctions by potential buyers with changing valuations and the seller. This problem arises in auctions of companies and asset sales, among other contexts. The bidder’s decision to approach the seller reveals that her valuation is high enough. In common-value auctions, such as battles between financial bidders, this revelation effect disincentivizes bidders from approaching the seller. In pure common-value auctions, no bidder ever approaches and auctions are seller-initiated. By contrast, in private-value auctions, such as battles between strategic bidders, the effect is the opposite, and equilibria often feature both seller- and bidder-initiated auctions. We link implications about the relation between the initiating party, bids, and auction outcomes to empirical evidence on auctions of companies.

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1. Introduction

Over the last several decades, auction theory has developed into a highly influential field with many important practical results.\(^1\) In particular, it has been extensively used to model transactions in the market for corporate control, such as mergers and acquisitions and intercorporate asset sales.\(^2\) To focus on the insights about the auction stage, with rare exceptions, the literature examines a situation when the asset is already up for sale.

In some cases, exogeneity of a sale is an innocuous assumption. For example, the U.S. Treasury auctions off bonds at a known frequency. In many cases, however, the decision to put the asset for sale is a strategic one. For example, the board of directors of a firm has a right but not an obligation to sell a division. Similarly, the decision of an art collector to sell an art piece is endogenous. An auction can be either bidder-initiated, when a potential bidder approaches the seller expressing an interest, which can lead the seller to auction the asset off, or seller-initiated, when the seller decides to auction the asset off without being approached by a potential buyer. To give a flavor of this heterogeneity, consider the following two recent deals in the M&A market. The acquisition of Taleo, a provider of cloud-based talent management solutions, by Oracle on February 9, 2012 for $1.9 billion is an example of a bidder-initiated auction. In January 2011, a CEO of a publicly traded technology company, referred in the deal background as Party A, contacted Taleo expressing an interest in acquiring it. Following this contact, Taleo hired a financial adviser that conducted an auction, engaging four more bidders. Oracle was the winning bidder, ending up acquiring Taleo. By contrast, the acquisition of Blue Coat Systems, a provider of Web security, by a private equity firm Thoma Bravo on December 9, 2011 for $1.1 billion is an example of a seller-initiated takeover auction. In early 2011, Elliot Associates, an activist hedge fund, amassed a 9% ownership stake in Blue Coat and forced its board to auction the company. Twelve bidders participated in the auction, and Thoma Bravo was the winner. Overall, there exists a considerable heterogeneity with respect to the initiator of the contest, which does not appear to be random. For example, acquisitions by strategic buyers are more likely to be bidder-initiated, while acquisitions by private equity firms are more likely to be target-initiated (Fidrmuc et al., 2012).\(^3\)

In this paper, we develop a theory of how potential buyers and the seller choose to initiate

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1. The formal analysis of auctions goes back to Vickrey (1961). The overview of results on auction theory can be found, for example, in Krishna (2010).
3. Initiation is also related to characteristics of the seller and auction outcomes (Masulis and Simsir, 2019).
auctions. In particular, we ask the following questions: Which characteristics of auctions and the economic environment determine whether auctions are bidder- or seller-initiated? How do bidding strategies and auction outcomes differ depending on how the auction was initiated? What are the implied inefficiencies and what are the potential remedies, if any?

To study these questions, we consider a dynamic framework, in which a seller owns an asset and faces two potential buyers. Each buyer has a signal about her valuation of the asset. Buyers' valuations may change over time. We do not assume that the auction takes place at an exogenous date and instead treat it as a strategic decision. The auction can be initiated by a bidder, when she approaches the seller, or by the seller when he chooses to auction the asset off without being approached by either bidder. The benefit of waiting for the seller is that with some likelihood, a bidder with a high valuation will appear and approach the seller, resulting in a higher expected price. Conversely, the benefit of selling without being approached is the lack of delay. We focus on stationary equilibria, i.e., when the distribution of signals, conditional on no auction in the past, stays the same over time.

The key driving force behind our results is that approaching the seller reveals that the valuation of the initiating bidder is sufficiently high. In a bidder-initiated auction, ex-ante identical bidders become endogenously asymmetric at the auction stage: the signal of the initiating bidder gets drawn from a more optimistic distribution. The other bidder uses this information to choose her bidding strategy and potentially re-value the asset. Similarly, the lack of an approaching bidder reveals information about valuations of all bidders: in a seller-initiated auction, each bidder knows that the valuation of the rival is sufficiently low, as she would have initiated the auction otherwise.

We show that the interplay between these information effects depends on the sources of bidders' valuations. In common-value auctions, e.g., when private equity firms compete to acquire a poorly-managed target, information effects discourage each bidder from approaching the seller. In pure common-value auctions, this effect is extreme: no bidder ever approaches, no matter how high her signal is. All auctions are initiated by the seller, if at all. By contrast, information effects work in the opposite direction in private-value auctions, e.g., when strategic bidders compete to purchase the asset they plan to integrate into their existing operations. Given the same signal, a bidder obtains a higher payoff in the auction she initiates than in an auction initiated by the rival.

The intuition behind these results is as follows. Consider a common-value setting: bidders have the same valuation of the asset but differ in their signals about it. Approaching the seller reveals information that the signal of the initiating bidder is sufficiently high: specifically, it is above a
certain cut-off. In turn, observing that the auction is bidder-initiated, the rival bidder updates her valuation of the asset upwards. As a result, she bids aggressively not only because she competes against a strong bidder but also because of her own higher valuation. In a pure common-value setting, this logic implies that the initiating bidder with the lowest signal among those that lead to initiation wins only when the rival’s signal is the lowest possible, in which case she pays the total value of the asset, obtaining no surplus. Such bidder would be better off waiting until either the rival bidder or the seller initiates the auction, as she would be able to get information rents then. Because the argument holds for any hypothetical equilibrium cut-off signal that leads to initiation, bidder-initiated auctions do not occur in equilibrium.

Next, consider a private-value setting and a bidder with a sufficiently high signal who contemplates approaching the seller. In contrast to the common-value setting, observing that the auction is bidder-initiated, the rival bidder does not update her valuation of the asset, which therefore remains low. As a result, she only bids aggressively because she competes against a strong bidder but not because of the valuation update. This logic implies that the bidder with a sufficiently high signal can take advantage of the rival’s low valuation by approaching the seller immediately: she always obtains positive surplus, even if her signal is the lowest among those that lead to initiation. In contrast, waiting until the rival initiates the auction ensures that the bidder will compete against a strong rival. Even though participating in a rival-initiated auction allows the bidder with a sufficiently high valuation to hide it, competing against a weak rival who adjusts her bid upwards is better than competing against a strong rival who adjusts her bid downwards. Thus, in contrast to the common-value setting, the bidder would be worse off waiting until the rival bidder initiates the auction, implying that bidder-initiated auctions can occur in equilibrium.

In the private-value framework, multiple equilibria often arise, because initiation decisions of bidders and the seller are interdependent. If bidders perceive a seller-initiated auction to be a very unlikely event, they will have strong incentives to initiate the auction, because, as described above, a rival-initiated auction makes them worse off and is likely to occur before the seller-initiated auction. In contrast, if bidders expect the seller to auction the asset off soon, they will have weak incentives to initiate the auction, because the seller-initiated auction makes them better off and is likely to occur before the rival-initiated auction. Intuitively, it is worthwhile to wait until the seller initiates the auction in this case because in contrast to the rival-initiated auction, it allows the bidder with a sufficiently high valuation to hide it without necessarily facing a strong rival.

Taken together, our results provide a benchmark with which one can compare empirical findings
on initiation of auctions. For example, our results are consistent with empirical evidence on target- and bidder-initiated strategic and private-equity deals: approximately 60% (35%) of strategic (private-equity) deals are initiated by the bidders (Fidrmuc et al, 2012). Our explanation of this difference is that financial, but not strategic, bidders have a large common value component in their valuations for targets. Our analysis also has implications about how bids and auction outcomes differ depending on whether auctions are bidder- or seller-initiated. For example, bidders bid more aggressively in a bidder-initiated auction than in a seller-initiated auction; in a bidder-initiated auction, conditional on the same valuation, a non-initiating bidder bids more aggressively than the initiating bidder, while unconditionally the initiating bidder bids more aggressively.

We extend the model to capture additional features of the market for corporate control, our lead application. Consider an inefficiently-run firm followed by potential bidders, whose valuations in this case are best represented by the common-value setting. Our results show that each bidder would be reluctant to initiate the auction to acquire such firm. If the firm’s management and board are entrenched, the seller would not initiate the auction either, which would result in the failure of the market for corporate control as a corporate governance mechanism. Our extensions lead to two results. First, while the market alone may be insufficient to resolve such inefficiencies, an activist investor, such as Elliot Associates in an earlier example, can use it to acquire a block of shares and force the firm to auction itself off. In this respect, shareholder activism and the market for corporate control are complements, rather than two different mechanisms for turning around poorly managed companies. Second, an interested bidder can acquire a toehold in the firm and ensure that she always obtains a positive surplus and is interested in initiation, implying that in a dynamic environment, the welfare effect of toeholds trades off allocative inefficiency of the auction and higher efficiency of bidder initiation.

Our paper belongs to the vast literature on auction theory. Virtually all of it only considers a stage when the auction takes place. Three exceptions are papers by Board (2007), Cong (2017), and Gorbenko and Malenko (2017), which also feature strategic timing of the auction. Board (2007) and Cong (2017) study the problem of a seller auctioning an option, such as the right to drill oil, where the timing of the sale and option exercise are decision variables. Gorbenko and Malenko (2017) assume that M&A contests are bidder-initiated and study the role of stock bids in alleviating bidders’ financial constraints. These papers do not study joint initiation by bidders and the seller and restrict attention to independent private values, so the issues examined in our paper do not arise.
Second, the paper is related to the literature that studies takeover contests as auctions. They have been modeled using both the common-value (e.g., Bulow, Huang, and Klemperer, 1999) and private-value framework (e.g., Fishman, 1988; Burkart, 1995; Povel and Singh, 2006). Like us, Bulow, Huang, and Klemperer (1999) interpret competition between strategic (or financial) bidders as a private-value (or common-value) auction. However, these papers do not study endogenous initiation of takeover contests. Our extension for shareholder activism relates to recent papers that study interactions between activism and the market for corporate control focusing on other aspects of the interaction – Burkart and Lee (2015) focus on the free-rider problem in tender offers, while Corum and Levit (2016) focus on the commitment problem of the bidder in a proxy fight.

Third, the paper is related to models of auctions with asymmetric bidders. Most literature on auction theory assumes that bidders are symmetric in the sense that their signals are drawn from the same distribution. Some recent literature (e.g., Maskin and Riley, 2000, 2003; Campbell and Levin, 2000; Lebrun, 2006; Kim, 2008) examines issues that arise when bidders are asymmetric. The novelty of our paper is that asymmetries at the auction stage are not assumed: they arise endogenously and are driven by incentives to approach the seller which differ with the bidder’s information. While bidders are ex-ante symmetric, at the auction stage they are not: the decision of one bidder to approach the previously unapproached seller makes it commonly known that bidders’ signals come from different partitions of the same ex-ante distribution of signals.

Finally, while on a different topic, the learning effect in common-value auctions is related to Ely and Siegel (2013). They develop a static model of firms, which are similar to our bidders, interviewing and hiring workers, which are similar to our sellers. Workers’ value added is common among firms with different signals, so a firm’s choice to interview an applicant, if publicly revealed, results in an update of other firms’ values, which in turn leads to unraveling where only the highest-ranked firm interviews the applicant. Because our model is dynamic, focuses on the optimal timing of the auction, features both bidder- and seller initiation, and compares common-value and private-value settings, most implications and results are quite different.

The remainder of the paper is organized as follows. Section 2 describes the setup of the model. Section 3 studies the common-value framework. Section 4 studies the private-value framework. Section 5 focuses on additional features of the market for corporate control, and discusses model

\[4\] Bulow and Klemperer (1996, 2009) provide motivations why running a simple auction is often a good way for the seller to sell the asset.

\[5\] Jiang, Li, and Mei (2016) and Boyson, Gantchev, and Shivdasani (2016) provide empirical evidence on interactions between shareholder activists and the market for corporate control.
assumptions. Section 6 lists empirical predictions. Section 7 concludes.

2. The Model Setup

The economy consists of one risk-neutral seller (male) and two potential risk-neutral buyers (female), indexed by $i = 1, 2$.

The seller has an asset for sale. In the context of application to mergers and intercorporate asset sales, the asset can be the whole company or a business unit. The seller’s valuation of the asset is normalized to zero. Time is continuous and indexed by $t \geq 0$.

At the initial date $t = 0$, each potential bidder $i \in \{1, 2\}$ randomly draws a private signal. Bidders’ signals are independent draws from the uniform distribution over $[0, s_0]$, where $s_0 \in [0, 1]$.

Conditional on all signals, the value of the asset to bidder $i$ is $v(s_i, s_{-i})$, where $s_{-i}$ is the signal of the rival bidder.

Assumption 1. Function $v(s_i, s_{-i})$ is continuous in both variables, strictly increasing in $s_i$, and satisfies $v(s_i, s_{-i}) \geq 0 \forall (s_i, s_{-i}) \in [0, 1]^2$.

Assumption 1 is standard. Continuity means that there are no gaps in possible valuations of the asset. Strict monotonicity in the first variable means that a higher private signal is always good news about the bidder’s valuation. Finally, $v(s_i, s_{-i}) \geq 0$ for any combination of signals means that bidders value the asset weakly more than the seller. This valuation structure follows the general symmetric model of Milgrom and Weber (1982). It covers two valuation structures commonly used in the literature:

- **The private-value framework:** $v(s_i, s_{-i}) = v(s_i)$. A bidder’s signal provides information only about her own valuation, but not about the valuation of her competitors.

- **The common-value framework:** $v(s_i, s_{-i}) = v(s_{-i}, s_i)$, which is symmetric in both variables. Conditional on both signals, bidders have the same valuation of the asset. However, bidders can differ in their assessments of it, because their private signals can be different.

We focus on these two valuation structures. There are two natural interpretations of common

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6The model can be extended to $N \geq 2$ bidders with the main qualitative effects intact.

7Because we assume a general functional form that maps signals into valuations, uniform distribution is, to a large extent, a normalization.

8Extending the model to the general case of private- and common-value mixes does not expand on the model’s economic insight.
versus private values in the context of auctions of companies and business units. The first interpretation deals with different types of bidders: we can interpret the common-value (private-value) auction as a battle between two financial (strategic) bidders. Intuitively, financial bidders use similar strategies after they acquire the target (i.e., have “common” value), but may have different estimates of potential gains (i.e., have different signals about the common value). In contrast, because synergies that strategic bidders expect to achieve from acquiring the target are often bidder-specific, they provide little information about valuation of the target to the other bidder.

The second interpretation deals with different types of targets rather than bidders. Broadly, value in an acquisition can be created either because the incumbent target management is inefficient or because the target and the acquirer have synergies that cannot be realized by the stand-alone acquirer. To the extent that inefficiency can be resolved by any bidder, acquisitions of the first type are common-value deals. At the same time, because synergies are bidder-specific, acquisitions of the second type tend to be private-value deals.

In practice, the environment changes over time, as either the business nature or management of a bidder or a target changes, or external economic shocks arrive. To capture this feature, as time goes by, each bidder can experience a sequence of shocks. A shock to each bidder arrives according to the Poisson process with intensity $\lambda > 0$, and shocks are independent across bidders. If bidder $i$ experiences a shock, she gets a new signal, $s_i'$, which is an independent draw from uniform distribution over $[0,1]$. Her valuation of the asset changes from $v(s_i, s_{-i})$ to $v(s_i', s_{-i})$, and that of the rival – from $v(s_{-i}, s_i)$ to $v(s_{-i}, s_{-i})'$.

Thus, the previous signal of the bidder becomes irrelevant once she experiences a shock. The general idea behind this assumption is that there is an option value of not acquiring the asset today if the valuation is positive but low.

The seller has the right to auction the asset off to the bidders at any time. In addition, each bidder has the right to approach the seller and initiate the auction at any time. We refer to the former type of auctions as *seller-initiated*, and to the latter type of auctions as *bidder-initiated*. Whether the auction is seller- or bidder-initiated is known to its participants. Once either party

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9 We define $v(s_i, s_{-i})$ as the discounted expected flow of utility to the bidder (e.g., the present value of discounted cash flows), which already incorporates possible future shocks.

10 Alternatively, first, we could assume that an existing bidder exits the game, obtains some exogenous exit payoff $X > 0$, and is replaced by a new bidder. Second, we could assume that shocks are mean-reverting. The first assumption would simplify the analysis. Both assumptions would not affect the results qualitatively.

11 This assumption can be justified as follows. The seller may voluntarily disclose whether the auction is bidder- or seller-initiated. In many contexts, the disclosed auction type can be verified ex post – for example, any public U.S. target is required to report the deal background as part of its SEC filings, and lying there has legal consequences. By the standard reasoning (Grossman, 1981; Milgrom, 1981), because it is common knowledge that the seller knows the auction type and this information is verifiable, he will always disclose it.
(the seller or one of the two bidders) initiates the auction, a sealed-bid first-price auction with no reserve price takes place. Specifically, each bidder simultaneously submits a bid to the seller in a concealed fashion. The two bids are compared, and the bidder with the highest bid acquires the asset and pays her bid. Once the asset is sold, the game is over. The winning bidder obtains the payoff that equals to her valuation less the price she pays. The losing bidder obtains zero payoff. The seller obtains the payoff that equals to the winning bid.

2.1 The equilibrium concept

The equilibrium concept is Markov Perfect Bayesian Equilibrium (MPBE). In the auction, the strategy of each bidder is a mapping from her own signal and the belief about the signal of the rival into a non-negative bid. Prior to the auction, the strategy of each bidder is a mapping from her own signal and the belief about the signal of the rival into the decision whether to approach the seller or wait. The strategy of the seller is a mapping from his belief about the signals of both bidders into the decision whether to auction the asset off or wait. Because bidders are ex-ante symmetric, we look for equilibria in which they follow symmetric strategies prior to the auction and symmetric bidding strategies in a seller-initiated auction. Furthermore, we look for equilibria in which at any time $t$ prior to the auction a bidder follows the cut-off strategy: she approaches the seller if and only if his signal is above some cut-off $\hat{s}_t$.

For the remainder of the paper, we consider the stationary case, defined as the situation in which the cut-off $\hat{s}_t$ stays constant over time at some level $\hat{s}$. This requires that the upper bound on the initial signal is $\hat{s}_0 = \hat{s}$. Because in practice there is often no clear starting date, at least, in the applications we look at, focusing on the stationary solution is reasonable. In Section 5.4 and the online appendix, we provide an analysis of the non-stationary dynamics, starting at $\hat{s}_0 = 1$.

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12 In Section 5.4, we discuss the practical motivation for our choice of the first- versus second-price auction, and how results would be affected in the latter.

13 The seller’s private valuation of the asset can also be important for his decision to offer it for sale. Lauermann and Wolinsky (2016) study common-value first-price auctions in which the seller obtains a private signal about his value and solicits a different number of bidders at a cost depending on the signal’s value. Being solicited thus discloses some information about the seller’s signal to bidders. Interestingly, this solicitation effect can result in non-competitive bids and inefficient information aggregation. While modeling the two-sided private information is beyond the scope of this paper, it is potentially interesting to examine interactions between seller and bidder initiation in the presence of the solicitation effect.

14 Because bidders with arbitrarily low signals always obtain arbitrarily low surplus from the auction, there is no equilibrium in which such bidders approach the seller while bidders with high signals do not. What is less clear, however, is whether there are equilibria in which bidders’ decisions are not described by a cut-off. Because the analysis of first-price auctions when distributions of valuations have an arbitrary number of gaps is, to our knowledge, an open problem, we cannot say anything about the possibility of such equilibria.
3. The Case of Common Values

In this section, we consider the case of pure common value, \( v(s_i, s_{-i}) = v(s_{-i}, s_i) \).

3.1 Equilibria in bidder- and seller-initiated auctions

First, we solve for the equilibrium at the auction stage.

3.1.1 A bidder-initiated auction

Consider a bidder-initiated auction with an exogenous cut-off type \( \hat{s} \). The equilibrium cut-off type is determined at the initiation stage. Denote the initiating and non-initiating bidder by \( I \) and \( N \). Then, from the point of view of bidder \( N \) (or bidder \( I \)) and the seller, the type of bidder \( I \) (or bidder \( N \)) is distributed uniformly over \( [s_I, \hat{s}] = [\hat{s}, 1] \) (over \( [s_N, \hat{s}] = [0, \hat{s}] \)). Thus, even though all bidders are ex-ante symmetric, initiation based on a cut-off type endogenously creates an asymmetry between them.

Conjecture that there is an equilibrium in pure strategies. Denote the equilibrium bid of bidder \( I \) and \( N \) with signal \( s \) by \( a_I(s; \hat{s}), s \) and \( a_N(s; \hat{s}), s \leq \hat{s} \), respectively, and conjecture that each is strictly increasing in \( s \) in the relevant range. Denote the corresponding inverses in \( s \) by \( \phi_I(b; \hat{s}) \) and \( \phi_N(b; \hat{s}) \). The expected payoffs of bidders \( I \) and \( N \) with signal \( s \) and bid \( b \) are

\[
\Pi_j(b, s, \hat{s}) = \mathbb{E}[v(s, x) - b|x \in [\underline{s}_k, \phi_k(b, \hat{s})]] = \int_{\underline{s}_k}^{\phi_k(b, \hat{s})} (v(s, x) - b) \frac{1}{\hat{s}_k - \underline{s}_k} dx, \tag{1}
\]

where \( j \neq k \in \{I, N\} \). The intuition behind the system of equations (1) is as follows. Consider the initiating bidder who bids \( b \). She wins the auction if and only if the bid of the non-initiating bidder is below \( b \), which happens if such bidder’s signal is below \( \phi_N(b, \hat{s}) \). Conditional on winning when the rival’s signal is \( x \in [0, \phi_N(b, \hat{s})] \), the value of the asset to the initiating bidder is \( v(s, x) \). Integrating over \( x \in [0, \phi_N(b, \hat{s})] \) yields (1) for \( j = I \). The same intuition explains the expected payoff of the non-initiating bidder. Taking the first-order conditions of (1), we obtain

\[
\frac{\partial \phi_j(b, \hat{s})}{\partial b} (v(s, \phi_j(b, \hat{s})) - b) - (\phi_j(b, \hat{s}) - \underline{s}_j) = 0 \tag{2}
\]

for \( j \in \{I, N\} \). The first and second terms of equations (2) represent the trade-off between the marginal benefit and the marginal cost of increasing a bid by a small amount. The marginal benefit is that bidder \( k \) wins a marginal event in which the signal of the rival bidder \( j \) is exactly
The marginal cost is that bidder \( k \) must pay more in case she wins. In equilibrium, \( b = a_j (s, \hat{s}) \) must satisfy (2) for \( j \in \{I, N\} \), implying \( s = \phi_j (b, \hat{s}) \). Plugging in and rearranging the terms, we obtain
\[
\frac{\partial \phi_j (b, \hat{s})}{\partial b} = \frac{\phi_j (b, \hat{s}) - \hat{s}_j}{v (\phi_k (b, \hat{s}), \phi_j (b, \hat{s})) - b}.
\]
(3)

The system of equations (3) is solved subject to the following boundary conditions. First, in equilibrium, the highest bid submitted by both bidders must be the same: \( a_j (\bar{s}_j, \hat{s}) \equiv \bar{a} (\hat{s}) \) for \( j \in \{I, N\} \). \( a_I (1, \hat{s}) > a_N (\hat{s}, \hat{s}) \) cannot occur in equilibrium, because then types of bidder \( I \) close enough to 1 would reduce their bids and still win the auction with probability 1. Similarly, \( a_I (1, \hat{s}) < a_N (\hat{s}, \hat{s}) \) cannot occur in equilibrium. Second, the lowest bid submitted by both bidders must be the same: \( a_j (\underline{s}_j, \hat{s}) \equiv \underline{a} (\hat{s}) \) for \( j \in \{I, N\} \). Suppose instead that \( a_I (\bar{s}, \hat{s}) > a_N (0, \hat{s}) \). From (1), for type \( \hat{s} \) of bidder \( I \) to get non-negative rents, \( a_I (\bar{s}, \hat{s}) \) cannot exceed \( \mathbb{E} [v (\bar{s}, x) | x \leq \phi_N (a_I (\bar{s}, \hat{s}), \hat{s})] \). Consider bidder \( N \) with type \( \phi_N (a_I (\bar{s}, \hat{s}), \hat{s}) > \phi_N (a_N (0, \hat{s}), \hat{s}) = 0 \), i.e., a bidder who bids exactly \( a_I (\hat{s}, \hat{s}) \). Her payoff is zero, because the initiating bidder never bids below \( a_I (\hat{s}, \hat{s}) \). However, if this bidder deviated to bidding \( b \in (a_I (\hat{s}, \hat{s}), v (\bar{s}, \phi_N (a_I (\hat{s}, \hat{s}), \hat{s}))) \), her payoff would be positive, because her bid would win with positive probability and, conditional on winning, the payoff would be positive, as \( b < v (\bar{s}, \phi_N (a_I (\hat{s}, \hat{s}), \hat{s})) \).

Because \( a_I (\bar{s}, \hat{s}) \leq \mathbb{E} [v (\bar{s}, x) | x \leq \phi_N (a_I (\bar{s}, \hat{s}), \hat{s})] < v (\bar{s}, \phi_N (a_I (\bar{s}, \hat{s}), \hat{s})) \) when \( \phi_N (a_I (\bar{s}, \hat{s}), \hat{s}) > 0 \), set \( (a_I (\bar{s}, \hat{s}), v (\bar{s}, \phi_N (a_I (\bar{s}, \hat{s}), \hat{s}))) \) is non-empty. Therefore, \( a_I (\bar{s}, \hat{s}) > a_N (0, \hat{s}) \) cannot occur in equilibrium. Similarly, \( a_I (\bar{s}, \hat{s}) < a_N (0, \hat{s}) \) cannot occur in equilibrium. Hence, the support of possible equilibrium bids for both bidders is given by \([\underline{a} (\hat{s}), \bar{a} (\hat{s})] \).

The upper boundary implies \( 1 = \phi_I (\bar{a} (\hat{s}), \hat{s}) \) and \( \bar{s} = \phi_N (\bar{a} (\hat{s}), \hat{s}) \). Consider the lower boundary \( \underline{a} (\hat{s}) \). First, it must be that \( \underline{a} (\hat{s}) \geq v (\bar{s}, 0) \), as otherwise either type \( \bar{s} \) of bidder \( I \) or type 0 of bidder \( N \) would find it optimal to deviate and submit a marginally higher bid. By doing this she can increase the probability of winning from zero to a positive number, and thus get a positive expected surplus instead of zero. Second, it must be that \( \underline{a} (\hat{s}) \leq v (\bar{s}, 0) \), otherwise a too high lower boundary would imply that low enough types get a negative surplus in equilibrium.

Thus, we have proved the following lemma:

**Lemma 1 (equilibrium in the bidder-initiated CV auction).** The equilibrium bidding strategies of the initiating and the non-initiating bidders, \( a_j (s, \hat{s}), j \in \{I, N\} \), are increasing
functions, such that their inverses satisfy (3), with boundary conditions

\[ 1 = \phi_I (\bar{a} (\hat{s}) , \hat{s}) , \quad \hat{s} = \phi_N (\bar{a} (\hat{s}) , \hat{s}) , \quad \hat{s} = \phi_I (v (\hat{s}, 0) , \hat{s}) , \quad 0 = \phi_N (v (\hat{s}, 0) , \hat{s}) . \]  \tag{4} \]

Example 1 in the appendix provides the closed-form solution for the case of additive valuations, \( v (s_i, s_{-i}) = \frac{1}{2} (s_i + s_{-i}) \), which is linear in a bidder’s signal. Figure 1 illustrates the equilibrium bids and expected bidder surpluses of Example 1 for the case \( \hat{s} = 0.5 \).

The equilibrium in the auction implies the payoff of type \( \hat{s} \) of bidder \( I \) is zero:

**Lemma 2.** The equilibrium payoff of the initiating bidder of type \( \hat{s} \) is zero.

**Proof.** \( \Pi_I (a (\hat{s}) , \hat{s}, \hat{s}) = 0 \) follows immediately from \( a (\hat{s}) = v (\hat{s}, 0) \).

The intuition behind the lemma is simple. Bidder \( N \) knows that bidder \( I \) approaches the seller if and only if her signal is at least \( \hat{s} \). Therefore, bidder \( N \) with signal \( s \) knows that the lowest possible valuation is \( v (s, \hat{s}) \geq v (0, \hat{s}) \). Similarly, bidder \( I \) with signal \( s \) knows that the lowest possible valuation is \( v (s, 0) \geq v (\hat{s}, 0) \). Because both bidders cannot possibly value the asset below \( v (\hat{s}, 0) \), no bidder bids less that this. In turn, bidder \( I \) with signal \( \hat{s} \) wins the auction only when bidder \( N \)’s signal is zero and at price \( a (\hat{s}) = v (\hat{s}, 0) \), leaving her with zero surplus. This argument holds for any cut-off type \( \hat{s} \).

The above argument extends the result of Engelbrecht-Wiggans, Milgrom, and Weber (1983) that a bidder with access to public information only always obtains zero surplus in equilibrium. An important difference is that bidder \( I \) here does retain private information, because her decision to approach the seller only reveals that her signal cannot be below \( \hat{s} \). Hence, bidder \( I \) with almost any signal \( s > \hat{s} \) obtains a positive expected surplus. This can be seen on the right panel of Figure 1. Only bidder \( I \) with the marginal signal, \( \hat{s} \), obtains zero surplus. In the common-value framework, to reveal her higher signal through a higher bid, the bidder must be compensated with a higher surplus, which takes the form of a higher probability of winning. However, bidder \( I \) with signal \( \hat{s} \) has no information rent: in equilibrium, the cut-off \( \hat{s} \) is known, so type \( \hat{s} \) has no lower types to separate from; hence, she does not get compensated with rents.
3.1.2 A seller-initiated auction

Suppose that the seller initiates the auction. Conditional on no bidder approaching the seller, all parties believe that each bidder’s signal is distributed uniformly over $[0, \hat{s}]$ for some cut-off signal $\hat{s}$. Because the auction is seller-initiated, bidders are symmetric, so the solution is standard (see, e.g., Chapter 6.4 in Krishna, 2010). We look for an equilibrium in symmetric bidding strategies. Denote the equilibrium bid by a bidder with signal $s$ by $a_S(s, \hat{s})$. Denote the corresponding inverse in $s$ by $\phi_S(b, \hat{s})$. The expected payoff of a bidder with signal $s$ and bid $b$ is

$$\Pi_S(b, s, \hat{s}) = \mathbb{E}[v(s, x) - b|x \in [0, \phi_S(b, \hat{s})]] = \int_{0}^{\phi_S(b, \hat{s})} (v(s, x) - b) \frac{1}{\hat{s}} dx. \quad (5)$$

The logic behind (5) and (1) is similar. A bidder with bid $b$ wins the auction if and only if the signal of the rival bidder is below $\phi_S(b, \hat{s})$. Conditional on winning when the rival’s signal is $x \in [0, \phi_S(b, \hat{s})]$, the value of the asset to the bidder is $v(s, x)$. Integrating over $x \in [0, \phi_S(b, \hat{s})]$ yields (5). Taking the first-order condition and using the fact that in equilibrium $b = a_S(s, \hat{s})$ (or, equivalently, $s = \phi_S(b, \hat{s})$), we obtain

$$\left(\frac{\partial a_S(s, \hat{s})}{\partial s}\right)^{-1} (v(s, s) - a_S(s, \hat{s})) - s = 0. \quad (6)$$

This equation is solved subject to the boundary condition $a_S(0, \hat{s}) = v(0, 0)$, or, equivalently, $0 = \phi_S(v(0, 0), \hat{s})$. Intuitively, a bidder with the lowest signal only wins against the rival with the lowest signal. Upon winning, she re-evaluates the asset to $v(0, 0)$. Because $v(0, 0)$ is the lowest possible asset value, bidders with the lowest signal bid exactly $v(0, 0)$ in equilibrium. These results lead to the following lemma:

**Lemma 3 (equilibrium in the seller-initiated CV auction).** The symmetric equilibrium bidding strategies of the non-initiating bidders, $a_S(s)$, are increasing functions that are independent of $\hat{s}$ and solve (6) with boundary condition $a_S(0) = v(0, 0)$. Specifically,

$$a_S(s) = \int_{0}^{s} v(x, x) \frac{1}{s} dx = \mathbb{E}[v(x, x)|x \leq s]. \quad (7)$$

In contrast to bidder $I$ in the bidder-initiated auction, bidders with all but the lowest signal 0 obtain positive expected payoff in the seller-initiated auction. In particular, the cut-off type $\hat{s}$ obtains a positive payoff. For the case of additive valuations, $v(s_i, s_{-i}) = \frac{1}{2} (s_i + s_{-i})$, $a_S(s)$ =
\[ E[x|x \leq s] = \frac{s}{2}. \]

### 3.2 The initiation game

Because the marginal type of the initiating bidder always obtains zero surplus in equilibrium, it is straightforward to show that pure common-value auctions are never bidder-initiated. Suppose, by contradiction, that \( \hat{s} < 1 \). Then, a bidder with a signal just above \( \hat{s} \) obtains a positive but infinitesimal surplus by initiating the auction. In contrast, if a bidder waits until either another party initiates the auction or her signal switches to a higher value, her expected surplus will be bounded away from zero. Thus, any \( \hat{s} < 1 \) is inconsistent with equilibrium. Because the seller does not expect a bidder to ever initiate, there is no value for him in delaying the auction. Thus, the auction is initiated by the seller with no delay:

**Proposition 1.** There exists a unique equilibrium cut-off \( \hat{s} = 1 \). Because no bidder initiates the auction, the seller initiates the auction immediately.

It is straightforward to extend the model by assuming that running an auction costs \( C > 0 \) to the seller. If \( C \) is below the expected revenues of the seller from the auction when both bidders’ signals are distributed uniformly over \([0, 1]\), then he initiates the auction at \( t = 0 \). In contrast, if \( C \) is above this value, then the sale never happens, as neither the seller nor the initiating bidder with a signal close to the cut-off benefits from it. In the application to auctions of companies, it is natural to interpret \( C \) as the degree of entrenchment of the management and board of the target: they will not contemplate a voluntary (seller-initiated) auction unless the expected revenues exceed \( C \). The following corollary summarizes the above results:

**Corollary 1.** Let \( \tilde{C} \equiv E \left[ a \left( \max_{i \in \{1, 2\}} s_i \right) \left| s_i \in [0, 1] \ \forall i \right. \right] \). If \( C < \tilde{C} \), the seller initiates the auction immediately at \( t = 0 \). If \( C > \tilde{C} \), the seller does not initiate the auction and no bidder approaches the seller.

### 4. The Case of Private Values

In this section, we consider the case of private values, \( v(s_i, s_{-i}) = v(s_i) \). In the following analysis, we impose the following natural restriction on equilibrium bids, which pins down the unique
equilibrium in the auction:

**Assumption 2.** No bidder bids above her valuation in equilibrium.

The rationale behind this assumption is that bidding above one’s valuation is a dominated strategy.\(^{15}\)

### 4.1 Equilibria in bidder- and seller-initiated auctions

First, we solve for the equilibrium at the auction stage.

#### 4.1.1 A bidder-initiated auction

Consider a bidder-initiated auction with a fixed cut-off type \( \hat{s} \). As before, denote the equilibrium bid of bidder \( I \) and \( N \) with signal \( s \) by \( a_I (s, \hat{s}) \), \( s \geq \hat{s} \) and \( a_N (s, \hat{s}), s \leq \hat{s} \), respectively. Denote their inverses in \( s \) by \( \phi_I (b, \hat{s}) \) and \( \phi_N (b, \hat{s}) \). The expected payoffs of bidders \( I \) and \( N \) with signal \( s \) and bid \( b \) are now

\[
\Pi_j (b, s, \hat{s}) = \mathbb{E} [v (s) - b | x \in [\hat{s}_k, \phi_k (b, \hat{s})]] = (v (s) - b) \frac{\phi_k (b, \hat{s}) - \hat{s}_k}{\hat{s}_k - \hat{s}_k},
\]

where \( j \neq k \in \{I, N\} \). Intuitively, bidder \( I \)'s (or bidder \( N \)'s) bid exceeds the bid of her rival with probability \( \phi_N (b, \hat{s}) \hat{s} \) (or \( \phi_I (b, \hat{s}) \hat{s} \)). Taking the first-order conditions of (8), we obtain

\[
\frac{\partial \phi_j (b, \hat{s})}{\partial b} (v (s) - b) - (\phi_j (b, \hat{s}) - \hat{s}_j) = 0
\]

for \( j \in \{I, N\} \). In equilibrium, \( b = a_j (s, \hat{s}) \) must satisfy (9), implying \( s = \phi_j (b, \hat{s}) \). Thus,

\[
\frac{\partial \phi_j (b, \hat{s})}{\partial b} = \frac{\phi_j (b, \hat{s}) - \hat{s}_j}{v (\phi_k (b, \hat{s})) - b}.
\]

The system of equations (10) is solved subject to the following boundary conditions. Similarly to the common-value case, the equilibrium maximum and minimum bids that win with a positive

\(^{15}\)As Kaplan and Zamir (2011) show, without this restriction, multiple equilibria in the first-price auction with asymmetric bidders arise, in which some bidders submit “non-serious” bids (i.e., bids that win with probability zero) above their valuations. Such equilibria are implausible, because even though “non-serious” bidders obtain zero surplus in equilibrium, a deviation by the rival results in their negative payoff. Thus, it is reasonable to rule out these strategies. Assumption 2 pins down the unique equilibrium in the auction (Lebrun, 2006).
probability, or “serious” bids of both bidders must be the same: \(a_j(\bar{s}, \hat{s}) \equiv \bar{\alpha}(\hat{s})\) and \(a_j(\check{s}, \hat{s}) \equiv \check{\alpha}(\hat{s})\). If the maximum bids are not the same, the bidder whose maximum bid is higher can increase her payoff by reducing her bid by a small amount: Doing so does not affect the probability of winning, which is one, and reduces the payment conditional on winning. If the minimum serious bid of bidder \(N\) is below that of bidder \(I\), bidder \(N\) never wins with such bid, which violates the definition of a serious bid. If the minimum bid of bidder \(I\) is below that of bidder \(N\), there must be discontinuity in the expected payoff of bidder \(N\) at the signal that results in the minimum serious bid. However, this would imply that bidder \(I\) with signals resulting in non-serious bids just below the minimum serious bid of bidder \(N\) would benefit from a deviation to such bid. Hence, the support of possible equilibrium bids for both bidders is \([\bar{\alpha}(\hat{s}), \check{\alpha}(\hat{s})]\).

The upper boundary implies \(1 = \phi_I(\bar{\alpha}(\hat{s}), \hat{s})\) and \(\hat{s} = \phi_N(\check{\alpha}(\hat{s}), \check{s})\). Next, the lowest type of bidder \(I\) submits the lowest serious bid: \(\check{s} = \phi_I(\check{\alpha}(\hat{s}), \hat{s})\). This lowest bid, in turn, determines the cut-off on the signal of bidder \(N\), who submits a serious bid: the cut-off is equal to the lowest bid. If the minimum bid is above such cut-off, bidder \(N\) with the cut-off signal would bid above her valuation, which would violate Assumption 2. If the minimum bid is below the cut-off, she would profitably deviate to increasing her bid by a small amount, which would result in a positive expected payoff, exceeding her equilibrium payoff of zero. Formally, \(v^{-1}(\check{\alpha}(\hat{s})) = \phi_N(\check{\alpha}(\hat{s}), \hat{s})\).

Assumption 2 uniquely pins down the minimum “serious” bid (Lebrun, 2006):

\[
\check{\alpha}(\hat{s}) = \arg \max_b \frac{v^{-1}(b)}{\hat{s}} (v(\hat{s}) - b) \quad \Rightarrow \quad \text{F.O.C.:} \quad \frac{v(\check{s}) - \check{\alpha}(\hat{s})}{v'(v^{-1}(\check{\alpha}(\hat{s})))} = v^{-1}(\check{\alpha}(\hat{s})). \tag{11}
\]

The following lemma summarizes the unique equilibrium in the bidder-initiated first-price auction. Existence and uniqueness results follow from Lebrun (2006):

**Lemma 4 (equilibrium in the bidder-initiated PV auction).** The equilibrium is unique (up to the non-serious bids of types \(s < v^{-1}(\check{\alpha}(\hat{s}))\) of non-initiating bidders). The equilibrium bidding strategies of the initiating and non-initiating bidders, \(a_j(s, \check{s})\), \(j \in \{I, N\}\), are increasing functions, such that their inverses satisfy (10), with boundary conditions

\[
1 = \phi_I(\bar{\alpha}(\hat{s}), \check{s}), \quad \check{s} = \phi_N(\check{\alpha}(\hat{s}), \check{s}), \quad \hat{s} = \phi_I(\check{\alpha}(\hat{s}), \hat{s}), \quad v^{-1}(\check{\alpha}(\hat{s})) = \phi_N(\check{\alpha}(\hat{s}), \hat{s}) \tag{12}
\]

and the lowest serious bid is given by (11).
Example 2 in the appendix provides the closed-form solution for the case \( v(s_i) = s_i \). Figure 2 illustrates the equilibrium bids and expected bidder surpluses for the case \( \hat{s} = 0.5 \).

Denote \( \Pi_j^* (s, \hat{s}) = \Pi_j (a_j (s, \hat{s}), s, \hat{s}), j \in \{I, N\} \). The next lemma shows that the payoff of the bidder with the cut-off signal, \( \hat{s} \), is higher if she is bidder \( I \) than bidder \( N \):

**Lemma 5.** For any \( \hat{s} \), \( \Pi_I^* (\hat{s}, \hat{s}) \geq \Pi_N^* (\hat{s}, \hat{s}) \). The inequality is strict if \( \hat{s} < 1 \).

This result is in stark contrast with the case of common value, in which bidder \( I \) with the cut-off signal always obtains zero expected payoff, which, in particular, is strictly lower than a payoff of bidder \( N \) with the same signal, resulting in unraveling in initiation. In the private-value framework, all else equal, the bidder with the cut-off signal is better off initiating the auction rather than being the non-initiating bidder. Intuitively, because bidders do not update valuations, the strength of competition is endogenous on who the initiator is and favors the initiator. If a bidder waits until the rival approaches the seller, she will compete against a rival with a strong signal (above \( \hat{s} \)). In contrast, if a bidder initiates the auction today, she will compete against a rival with a weak signal (below \( \hat{s} \)). In turn, the rival adjusts her bid upwards (or downwards) upon the bidder’s (or rival’s) initiation compared to the symmetric bidder case, believing that she competes against a strong (or weak) bidder. However, this adjustment, in the absence of valuation updating, is second-order on the bidder’s profits and cannot eliminate the benefit of being the initiator.

### 4.1.2 A seller-initiated auction

Suppose that the seller initiates the auction. Conditional on no bidder approaching the seller, all parties believe that each bidder’s signal is distributed uniformly over \([0, \hat{s}]\). Bidders are symmetric, so the solution is standard (see, e.g., Krishna, 2010). Denote the equilibrium bid by a bidder with signal \( s \) by \( a_S (s, \hat{s}) \) and the corresponding inverse in \( s \) by \( \phi_S (b, \hat{s}) \). The expected payoff of a bidder with signal \( s \) and bid \( b \) is

\[
\Pi_S (b, s, \hat{s}) = \mathbb{E} [v(s) - b | x \in [0, \phi_S(b, \hat{s})]] = (v(s) - b) \frac{\phi_S(b, \hat{s})}{\hat{s}}. \tag{13}
\]

17
Taking the first-order condition and using the fact that in equilibrium, \( b = a_S(s, \hat{s}) \) (or, equivalently, \( s = \phi_S(b, \hat{s}) \)), we obtain

\[
\left( \frac{\partial a_S(s, \hat{s})}{\partial s} \right)^{-1} (v(s) - a_S(s, \hat{s})) - s = 0.
\]

This equation is solved subject to the boundary condition \( a_S(0, \hat{s}) = v(0) \). The following lemma summarizes the equilibrium:

**Lemma 6 (equilibrium in the seller-initiated PV auction).** The symmetric equilibrium bidding strategies of the non-initiating bidders, \( a_S(s) \), are increasing functions that are independent of \( \hat{s} \) and solve (14) with boundary condition \( a_S(0) = v(0) \). Specifically,

\[
a_S(s) = \int_0^s v(x) \frac{1}{s} dx = \mathbb{E}[v(x)|x \leq s].
\]

For the case \( v(s_i) = s_i \), \( a_S(s) = \mathbb{E}[x|x \leq s] = \frac{s}{2} \).

Denote \( \Pi^*_S(s, \hat{s}) = \Pi_S(a_S(s), s, \hat{s}) \). The next lemma shows that a seller-initiated auction leads to a higher expected payoff to bidder \( I \) with signal \( \hat{s} \) than an auction initiated by her:

**Lemma 7.** For any \( \hat{s} \), \( \Pi^*_S(\hat{s}, \hat{s}) \geq \Pi^*_I(\hat{s}, \hat{s}) \). The inequality is strict if \( \hat{s} > 0 \).

The intuition behind Lemma 7 is as follows. Consider a bidder with signal \( \hat{s} \). If the auction is initiated by the seller, the rival believes that the bidder’s signal is weak (below \( \hat{s} \)). In contrast, if the auction is initiated by the bidder, the rival believes that her signal is strong (above \( \hat{s} \)). In response, the rival adjusts her bid upwards upon the bidder’s initiation compared to the symmetric bidder case. In turn, the seller-initiated auction leaves a higher expected payoff to the bidder.

Together, Lemmas 5 and 7 imply that incentives of a bidder to approach the seller depend on whether her best outside option is to wait for another bidder to approach or for the seller to put the asset up for sale. When a bidder expects the seller to sell soon, she benefits from waiting. In the extreme case of an immediate sale, no bidder approaches the seller, as \( \Pi^*_S(\hat{s}, \hat{s}) > \Pi^*_I(\hat{s}, \hat{s}) \) for any \( \hat{s} > 0 \). When a bidder expects the rival to approach the seller soon, she benefits from initiating the deal herself. A practical implication for the market of distressed assets is as follows. Bidders are reluctant to approach the seller when they expect him to put the asset up for sale soon regardless of the demand for it, such as when the seller is close to bankruptcy. This intuition
holds regardless of whether the asset is commonly or privately valued by market participants.

4.2 The initiation game

Having solved for the equilibria in bidder- and seller-initiated auctions for any cut-off \( \hat{s} \), we next look for equilibrium cut-offs \( \hat{s} \). We first solve a bidder’s problem taking the initiation strategy of the seller and the other bidder as given. Applying the symmetry condition, we obtain the equilibrium initiation strategy of both bidders for any given initiation strategy of the seller. Then, we solve the seller’s problem taking the equilibrium strategy of bidders as given and combine the two to obtain equilibria.

4.2.1 A bidder’s problem

Recall that we focus on the stationary case where the distribution of bidders’ signals, conditional on no auction having taken place, is uniform over \([0, \hat{s}]\) for some \( \hat{s} \). Stationarity and the restriction to Markov strategies imply that the initiation strategy of the seller is the same at any time \( t \).

Let \( \mu dt \) denote the probability with which the seller initiates the auction during any short period of time \((t, t + dt)\), \( \mu \in [0, \infty] \). Here, \( \mu = 0 \) means that the seller never initiates the auction; \( \mu = \infty \) means that the seller initiates the auction over the next instant with probability one; and \( \mu \in (0, \infty) \) means that the seller randomizes between initiating the auction and waiting.

We fix \( \mu \) and solve for the symmetric equilibrium initiation strategy of bidders. Suppose that a bidder believes that the rival approaches the seller if and only if her signal exceeds \( \hat{s} \). Consider the bidder with signal \( s \). Denote the expected continuation value of this bidder by \( V_B(s, \hat{s}, \mu) \). This value satisfies

\[
V_B(s, \hat{s}, \mu) = \max \left\{ \Pi^*_I(s, \hat{s}), \frac{\lambda (1 - \hat{s}) \Pi^*_N(s, \hat{s}) + \mu \Pi^*_S(s, \hat{s}) + \lambda X(\hat{s})}{r + \lambda (1 - \hat{s}) + \lambda + \mu} \right\},
\]

where \( X(\hat{s}) \) is the bidder’s expected continuation value when she experiences a shock: \( X(\hat{s}) = \int_0^1 V_B(s', \hat{s}, \mu) ds' \). The intuition behind (16) is as follows. The continuation value is the maximum of the expected payoff from immediate initiation, which yields \( \Pi^*_I(s, \hat{s}) \), and waiting, which yields the second term of (16). Three independent events can occur if the bidder chooses to wait. First, with intensity \( \lambda (1 - \hat{s}) \), a rival with signal above \( \hat{s} \) appears and approaches the seller, and the bidder obtains \( \Pi^*_N(s, \hat{s}) \). Second, with intensity \( \mu \), the seller puts the asset up for sale, and the bidder obtains \( \Pi^*_S(s, \hat{s}) \). Third, with intensity \( \lambda \), the bidder experiences a shock and obtains \( X(\hat{s}) \).
By continuity of \( \Pi_I^s(\cdot), \Pi_N^s(\cdot), \) and \( \Pi_S^s(\cdot) \) in \( s \), the cut-off type must satisfy
\[
\Pi_I^s(\hat{s}, \hat{s}) = \frac{\lambda (1-\hat{s}) \Pi_N^s(\hat{s}, \hat{s}) + \mu \Pi_S^s(\hat{s}, \hat{s}) + \lambda X(\hat{s})}{r + \lambda (1-\hat{s}) + \lambda + \mu} \implies (17)
\]

This transformed equation is intuitive. For the indifferent type \( \hat{s} \), the cost of waiting equals the benefit. The cost of waiting (the left-hand side) is due to the delay in the profit from the bidder-initiated auction and the relative loss from a potential rival-initiated auction. The benefit of waiting (the right-hand side) is the relative benefit from a potential seller-initiated auction and from the option value of obtaining a higher valuation.

Existence of the indifferent type \( \hat{s} \), given by (17), is only a necessary condition for existence of a stationary equilibrium with bidder-initiated auctions. We also need to verify that if type \( \hat{s} \) is indifferent, then all types above \( \hat{s} \) find it optimal to approach the seller, while all types below \( \hat{s} \) find it optimal to wait. The next proposition shows that this condition holds if and only if \( \mu \) is below some cut-off level, denoted by \( \hat{\mu}(\hat{s}) \):

**Proposition 2.** Fix \( \mu \geq 0 \) and let \( \hat{s} < 1 \) denote the equilibrium cut-off signal for bidder-initiated auctions. Then, \( \hat{s} \) satisfies (17) and
\[
\mu < \hat{\mu}(\hat{s}) \equiv \frac{v^{-1}(a(\hat{s}))}{\hat{s} - v^{-1}(a(\hat{s}))} (r + \lambda) - \lambda (1 - \hat{s}).
\]

Any \( \hat{s} \) that satisfies (17) and (18) corresponds to a stationary equilibrium with threshold \( \hat{s} \).

### 4.2.2 The seller’s problem

Next, we fix \( \hat{s} \) and consider the seller’s choice between initiating the auction and waiting for the bidder to initiate it. If the seller waits, his payoff is \( \frac{2\lambda(1-\hat{s})}{r + 2\lambda(1-\hat{s})} R_B(\hat{s}) \), where \( R_B(\hat{s}) \) is the expected revenue in the bidder-initiated auction. If the seller initiates the auction immediately, his expected revenue is \( R_S(\hat{s}) \). \( R_B(\hat{s}) \) and \( R_S(\hat{s}) \) admit simple integral representations:
\[
R_S(\hat{s}) = \int_0^{a_S(\hat{s})} bd \left( \frac{\phi_S(b, \hat{s})}{\hat{s}} \right)^2, \quad R_B(\hat{s}) = \int_{a(\hat{s})}^{\hat{a}(\hat{s})} bd \left( \frac{\phi_I(b, \hat{s}) - \hat{s} \phi_N(b, \hat{s})}{1 - \hat{s}} \right).
\]

Because \( R_B(\hat{s}) > R_S(\hat{s}) \), the seller faces the trade-off between delay and lower expected revenues. The next proposition summarizes the best response of the seller:
Proposition 3. Fix \( \hat{s} \). The seller’s optimal strategy is: (a) to never initiate the auction \( (\mu = 0) \) if \( \frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B(\hat{s}) \geq R_S(\hat{s}) \); (b) to initiate the auction immediately \( (\mu = \infty) \) if \( \frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B(\hat{s}) \leq R_S(\hat{s}) \); (c) to randomize between initiating the auction and waiting (any \( \mu \in (0, \infty) \)) if \( \frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B(\hat{s}) = R_S(\hat{s}) \).

4.2.3 Equilibria

Combining the previous derivations, we can characterize the set of equilibrium cut-offs \( \hat{s} \). Because initiation decisions of bidders are strategic complements, there can be multiple equilibria. If a bidder expects other bidders to only approach the seller if \( s \) is very high (i.e., \( \hat{s} \uparrow 1 \)), then payoffs of the initiating and non-initiating bidder with the cut-off signal are close. In turn, a bidder has weak incentives to approach the seller. In contrast, if \( \hat{s} \) sufficiently low so that \( \Pi_I^* (\hat{s}, \hat{s}) - \Pi_N^* (\hat{s}, \hat{s}) \) is high, a bidder has strong incentives to approach. Note that this argument is the opposite of that in the common-value model. There, initiation decisions of bidders are strategic substitutes: if a bidder expects her rival to approach the seller, she has weak incentives to approach herself. The next proposition shows the existence of at least one equilibrium:

Proposition 4 (equilibrium with only seller-initiated auctions). There always exists an equilibrium in which all auctions are seller-initiated: \( \hat{s} = 1 \) and \( \mu = \infty \).

Intuitively, if the seller believes that bidders will not approach, then delaying the seller-initiated sale is costly. Similarly, if bidders believe that the seller will put the asset up for sale immediately, approaching the seller earlier is costly.

To have both seller- and bidder-initiated auctions in equilibrium at the same time, the seller must play mixed strategies. Equivalently, the seller must be indifferent between initiating the auction himself and waiting until he is approached by a bidder. To characterize such equilibria, we further transform (17) to eliminate the continuation value in \( X(\hat{s}) \). Integrating (16) over \([0, 1]\), solving for \( X(\hat{s}) \), and inserting this solution into (17), we obtain

\[
\begin{align*}
r \left( \Pi_I^* (\hat{s}, \hat{s}) + D(\hat{s}, \mu) \Pi_I^* (\hat{s}) \right) + \lambda (1 - \hat{s}) \left( \Pi_I^* (\hat{s}, \hat{s}) - \Pi_N^* (\hat{s}, \hat{s}) \right) + D(\hat{s}, \mu) \left( \Pi_I^* (\hat{s}) - \Pi_N^* (\hat{s}) \right) & \\
= \mu \left( \Pi_S^* (\hat{s}, \hat{s}) - \Pi_I^* (\hat{s}, \hat{s}) + D(\hat{s}, \mu) \left( \Pi_S^* (\hat{s}) - \Pi_I^* (\hat{s}) \right) \right) + \lambda \left( \Pi_I^* (\hat{s}) - \Pi_I^* (\hat{s}, \hat{s}) \right),
\end{align*}
\]

(20)
where \( D(\hat{s}, \mu) = \frac{\lambda \hat{s}}{r + 2\lambda(1 - \hat{s}) + \mu} \), \( \bar{\Pi}_I^s (\hat{s}) \) and \( \bar{\Pi}_N^s (\hat{s}) \) are the average (across signals) expected payoffs of bidders \( I \) and \( N \) in a bidder-initiated auction, and \( \bar{\Pi}_S^s (\hat{s}) \) is the average expected payoff of a bidder in a seller-initiated auction. In the appendix we show that \( \bar{\Pi}_I^s (\hat{s}), \bar{\Pi}_N^s (\hat{s}), \) and \( \bar{\Pi}_S^s (\hat{s}) \) are given by relatively simple expressions. This transformation leads to the following result:

**Proposition 5 (equilibria with seller- and bidder-initiated auctions).** Consider \( \hat{s} < 1 \) and \( \mu > 0 \). There exists an equilibrium with cut-off \( \hat{s} \), in which the seller initiates the auction with intensity \( \mu dt \), if and only if \( \mu \) and \( \hat{s} \) satisfy (20), \( \mu < \bar{\mu} (\hat{s}) \), and \( \frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B (\hat{s}) = R_S (\hat{s}) \).

The final condition of Proposition 5 states that the seller is indifferent between own and bidder’s initiation. It is independent of \( \mu \). Thus, it determines the set of potential equilibrium \( \hat{s} \). Given each \( \hat{s} \), (20) determines the potential equilibrium frequency \( \mu \equiv \mu(\hat{s}) \) of seller initiation. If it is below \( \bar{\mu} (\hat{s}) \), then pair \((\hat{s}, \mu)\) constitutes a stationary equilibrium.

Finally, there can be an equilibrium in which all auctions are bidder-initiated:

**Proposition 6 (equilibrium with only bidder-initiated auctions).** There exists an equilibrium with cut-off \( \hat{s} \), in which the seller never initiates the auction if and only if \( \hat{s} \) satisfies (20) evaluated at \( \mu = 0 \) and \( \frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_B (\hat{s}) \geq R_S (\hat{s}) \).

The final condition of Proposition 6 states that for the seller, waiting for the bidder to approach is at least weakly preferred to immediate seller initiation.

Example 3 in the appendix provides the quasi-closed form solution for the case \( v(s_i) = s_i \). Figure 3 illustrates equilibrium \((\hat{s}, \mu)\) with seller- and bidder-initiated auctions for the case \( r = 0.05, \lambda = 0.5 \) (three in total, \((0.635, 0), (0.950, 0.256), \) and \((1, \infty))\).

Our analysis of the base model shows that initiation of private- and common-value auctions can be drastically different. While both auctions, in equilibrium, can be initiated by sellers, only private-value auctions can be initiated by bidders. Furthermore, as Figure 3 illustrates, in the private-value setting the initiation game can have multiple equilibria: auctions can be only bidder-initiated or only seller-initiated, or initiated by either party. This result suggests that different markets with otherwise similar characteristics can feature very different asset sale initiation patterns and, as a consequence, different allocation and welfare properties.
5. Extensions and Discussion

The base model is designed to explain broad initiation patterns of endogenous auctions with our lead application being the market for corporate control. In this section, we discuss several extensions of our model that capture additional features of this market. The goal is to study whether and how these additional features can alleviate the problem that bidders are not willing to approach sellers in common-value auctions.

5.1 Shareholder activists as facilitators of auctions of companies

As Corollary 1 shows, if a company is underperforming, inducing common value in potential bidders, and its management is entrenched, such company can remain without a change in ownership for a long time, even when the value added from the change is substantial. Thus, the role of takeovers as a corporate governance mechanism can be limited.

The lack of incentive for bidders to initiate common-value auctions gives rise to alternative ways of promoting takeovers, in particular, to shareholder activism. We extend our base model by adding an activist investor, and show how it can be beneficial in facilitating auctions of companies. Suppose that there is an activist that arrives to the market (e.g., by discovering the target) with intensity $\lambda_A > 0$. After it arrives, the activist can submit an order to buy fraction $\alpha$ of the target. In addition, there is a liquidity trader that arrives with intensity $\lambda_L > 0$ and submits an order to buy fraction $\alpha$ of the target.\(^{16}\) If an activist buys fraction $\alpha$, it can undertake an activism campaign at cost $A > 0$, which results in putting the target up for sale. Assume that a competitive market-maker prices the block at its expected value conditional on receiving the order. In this setup, the activist cannot create value outside of the campaign, so it can only be optimal to buy fraction $\alpha$ to subsequently undertake the campaign. Suppose that upon discovering the target, the activist buys fraction $\alpha$ with probability $q$. Then, the price of fraction $\alpha$ must be\(^{17}\)

$$
\alpha P(q) = \alpha R_S(1) \frac{\lambda_A q}{\lambda_A q + \lambda_L} \left( 1 + \frac{\lambda_L}{r + \alpha A q} \right). \tag{21}
$$

Intuitively, the activist invests with intensity $\lambda_A q$. Hence, an order of fraction $\alpha$ comes from it with probability $\frac{\lambda_A q}{\lambda_A q + \lambda_L}$ and from the liquidity trader with probability $\frac{\lambda_L}{\lambda_A q + \lambda_L}$. In the former case, the target value is $R_S(1)$, the expected seller revenue in the absence of bidder initiation. In

\(^{16}\) Here, $\alpha$ is an exogenous parameter but it can be endogenized in a richer model.

\(^{17}\) See the appendix for the derivation.
the latter case, the activist has not arrived yet, so the target value is a discounted value of $R_S(1)$.

The activist’s payoff from undertaking a campaign, net of cost of acquiring fraction $\alpha$, is

$$\alpha R_S(1) - \alpha P(q) - A = \alpha R_S(1) \frac{r \lambda_L}{(q \lambda_A + \lambda_L)(r + q \lambda_A)} - A.$$  \hspace{1cm} (22)

Equation (22) is strictly decreasing in $q$. Hence, if (22) is non-negative at $q = 1$, then the activist buys fraction $\alpha$ with probability one and undertakes the campaign. If (22) is negative at $q = 1$, then the activist buys fraction $\alpha$ with probability $q \in (0, 1)$ at which (22) equals zero, and then undertakes the campaign. The next proposition summarizes the equilibrium:

**Proposition 7.** In equilibrium, upon discovering the target, the activist acquires the block with probability $q$, given by: (a) $q = 1$ if $A \leq \tilde{A} \equiv \frac{r \lambda_L \alpha R_S(1)}{(r + \lambda_L)(r + q \lambda_A)}$; (b) $q = \frac{\sqrt{(r + \lambda_L)^2 + 4 \lambda_L \alpha R_S(1) - r + \lambda_L}}{2 \lambda_A}$ if $A \in (\tilde{A}, \tilde{A})$; (c) $q = 0$ if $A \geq \tilde{A} \equiv \alpha R_S(1)$. The implied price of the block is given by (21).

This analysis implies that shareholder activism and the market for corporate control are not two unrelated mechanisms for disciplining the management but rather complement each other: activists use the market for corporate control to facilitate transactions of targets, inefficiencies in which would not be corrected otherwise.

5.2 Toeholds

In common-value auctions of companies, the initiating bidder with the cut-off signal can obtain positive rents if it secretly acquires a toehold in the target prior to initiating the auction. We extend our base model to allow for endogenous toehold acquisition.

Consider a common-value framework. Suppose that a bidder who considers approaching the target can submit an order to acquire fraction $\alpha$ of the target without disclosing her intent. In addition, there is a liquidity trader that arrives with intensity $\lambda_L > 0$ and submits an order to buy fraction $\alpha$ of the target. A competitive market-maker prices the block at its expected value conditional on receiving the order. This setup captures, in reduced form, the U.S. practice that a bidder may acquire up to 5% of a company’s outstanding shares secretly, beyond which she is required to publicly file Schedule 13(D). For simplicity, assume that the holding cost of the target’s shares is infinite, i.e., a bidder never acquires a toehold unless she approaches the seller immediately after. To simplify the exposition, we focus on the case in which the seller
does not initiate the auction (see the appendix for the generalization). By analogy with (21), the equilibrium price of fraction $\alpha$ must be\footnote{See the derivation in the appendix.}

$$
\alpha P = \alpha R_B (\hat{s}) \frac{2 \lambda (1 - \hat{s})}{2 \lambda (1 - \hat{s}) + \lambda_L} \left( 1 + \frac{\lambda_L}{r + 2 \lambda (1 - \hat{s})} \right).
$$

(23)

Intuitively, each bidder initiates the auction with intensity $\lambda (1 - \hat{s})$. Hence, an order of fraction $\alpha$ comes from one of the bidders with probability $\frac{2 \lambda (1 - \hat{s})}{2 \lambda (1 - \hat{s}) + \lambda_L}$ and from the liquidity trader with probability $\frac{\lambda_L}{2 \lambda (1 - \hat{s}) + \lambda_L}$. In the former case, the target value is $R_B (\hat{s})$, the expected seller revenue from the bidder-initiated auction in the presence of the toehold. In the latter case, the interested bidder has not arrived yet, so the target value is a discounted value of $R_B (\hat{s})$.

Consider a bidder-initiated auction when bidder $I$ has toehold $\alpha$. The expected payoffs of the bidders $I$ and $N$ with signal $s$ and bid $b$ are

$$
\Pi_j (b, s, \hat{s}) = \int_{\phi_k (b, \hat{s})}^{\phi_k (b, s)} (v (s, x) - (1 - \alpha) b) \frac{1}{s_k - s} dx + \int_{\phi_k (b, \hat{s})}^{\phi_k (b, s)} \alpha_j a_k (x, \hat{s}) \frac{1}{s_k - s} dx,
$$

(24)

where $j \neq k \in \{I, K\}$ and $(\alpha_I, \alpha_N) = (\alpha, 0)$. For intuition, consider bidder $I$. The first term in (24) captures the payoff of bidder $I$ upon winning. In this case, bidder $I$ acquires the remaining $1 - \alpha$ shares at bid $b$, and obtains asset value $v (s, x)$. The second term in (24) captures the payoff of bidder $I$ upon losing. In this case, bidder $I$ sells fraction $\alpha$ to the rival at her bid $a_N (s, \hat{s})$. Bidder $N$ only obtains the payoff if she wins. Taking the first-order conditions of (24), we obtain

$$
\frac{\partial \phi_j (b, \hat{s})}{\partial b} (v (s, \phi_j (b, \hat{s})) - (1 - \alpha_k) b - \alpha_k a_j (\phi_j (b, \hat{s}), \hat{s})) - (1 - \alpha_k) (\phi_j (b, \hat{s}) - \hat{z}_j) = 0
$$

(25)

for $j \neq k \in \{I, N\}$. Compared to (2) in the model without toeholds, (25) includes an additional term. In equilibrium, $b = a_j (s, \hat{s})$ must satisfy (25) for $j \in \{I, N\}$, implying $s = \phi_j (b, \hat{s})$. Also, note that $a_j (\phi_j (b, \hat{s}), \hat{s}) = b$. Plugging in and rearranging the terms, we obtain

$$
\frac{\partial \phi_j (b, \hat{s})}{\partial b} = \frac{(1 - \alpha_k) (\phi_j (b, \hat{s}) - \hat{z}_j)}{v (\phi_k (b, \hat{s}), \phi_j (b, \hat{s})) - b}.
$$

(26)

The system of equations (26) is solved subject to the following boundary conditions: $a_j (\hat{s}, \hat{s}) \equiv \tilde{a} (\hat{s})$ and $a_j (\hat{z}_j, \hat{s}) \equiv a (\hat{s})$, so that the support of possible equilibrium bids for both bidders is $[a (\hat{s}), \tilde{a} (\hat{s})]$. In turn, the equilibrium inverse bidding functions solve (26) subject to boundary
The payoff of bidder $I$ with signal $\hat{s}$ from the auction, net of the cost of acquiring toehold, is

$$
\Pi_I^*(\hat{s}, \hat{s}) - \alpha P = \alpha \left( \int_{0}^{\hat{s}} a_N(x, \hat{s}) \frac{1}{\hat{s}} dx - R_B(\hat{s}) \frac{2\lambda (1 - \hat{s})}{2\lambda (1 - \hat{s}) + \lambda_L} \left( 1 + \frac{\lambda_L}{r + 2\lambda (1 - \hat{s})} \right) \right),
$$

where $\Pi_I^*(\hat{s}, \hat{s}) = \Pi_I(a(\hat{s}), \hat{s}, \hat{s})$ and $R_B(\hat{s}) = \mathbb{E} \left[ \max \{ a_I(s_1, \hat{s}), a_N(s_2, \hat{s}) \} \right]$. A necessary condition for a bidder to acquire a toehold and initiate the auction is that (27) is positive. Condition (27) is never satisfied if $\hat{s}$ is low enough and always satisfied if $\hat{s}$ is high enough. Intuitively, if too many types are expected to initiate, the price of a toehold is high, so it is too expensive for the bidder with signal $\hat{s}$. Thus, factors that increase the price of a toehold also limit bidder initiation.

For an equilibrium, a bidder with signal $\hat{s}$ must be indifferent between approaching the seller and waiting. The cut-off signal is determined from the indifference condition (17) computed at $\mu = 0$. As in Proposition 3, the seller never initiates the auction if $\frac{2\lambda (1 - \hat{s})}{r + 2\lambda (1 - \hat{s})} R_B(\hat{s}) \geq R_S(\hat{s})$.

Example 4 in the appendix provides a quasi-closed form solution for the case $v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i})$. For $r = 0.05$, $\lambda = 0.5$, and $\lambda_L = 0.5$, the equilibrium with $\mu = 0$ has $\hat{s} = 0.934$.

The implication of this analysis is that while toeholds are often considered to be a source of inefficiency, because in a static model they can result in the acquisition of an asset by a bidder with a lower valuation, they help bidders initiate positive-value deals that would not occur otherwise in our dynamic setting. Because toeholds can be valuable, it is important to further study optimal disclosure requirements of blocks of shares in a dynamic setting.

### 5.3 Preemption and participation costs

The base model assumes that upon initiation of the auction, both bidders enter it. In practice, participating in the auction can be costly forcing some bidders to avoid the auction. Uncontested bidders, in turn, are left with a higher surplus. We extend our base model to include participation costs and examine their impact on initiation patterns in the common-value framework.

Suppose that a bidder has to pay a cost $C$ to participate in the auction, which cannot be reimbursed by the seller. We later relax this assumption. First, we show that equilibria, in which non-initiating bidders enter a bidder-initiated auction, do not exist. Suppose that $\hat{s}_N$ is an equilibrium cut-off signal, at or above which bidder $N$ enters a bidder-initiated auction. Upon this event, bidder $I$ re-values the asset to $v(s_1, \hat{s}_N) \geq v(\hat{s}, \hat{s}_N)$. Then, no bidder pays less than $v(\hat{s}, \hat{s}_N)$ in equilibrium. In turn, bidder $N$ with signal $\hat{s}_N$ wins the auction only when bidder $I$’s
signal is \( \hat{s} \), so that she submits \( v(\hat{s}_N, \hat{s}) \). Upon winning, bidder \( N \) re-values the asset to \( v(\hat{s}_N, \hat{s}) \), leaving her without surplus and with negative net surplus once participation costs are accounted for. Such bidder therefore has an incentive not to participate. This argument holds for any cut-off type \( \hat{s}_N \), so the only possible equilibrium is the one in which bidder \( I \) is uncontested. In this case, bidder \( I \) bids the lowest possible value, \( a(s, \hat{s}) = v(0, 0) \), and obtains the expected net payoff of

\[
\Pi_I(a(s, \hat{s}), s, \hat{s}) - C = \mathbb{E}[v(s, x) - v(0, 0)|x \leq \hat{s}] - C = \int_0^{\hat{s}} (v(s, x) - v(0, 0)) \frac{1}{\hat{s}} dx - C, \tag{28}
\]

while bidder \( N \) obtains zero equilibrium payoff. Second, suppose that \( \hat{s}_S \) is an equilibrium cut-off signal, at or above which bidders enter a seller-initiated auction. Similarly to bidder-initiated auctions, if the rival does not participate, the participating bidder bids \( v(0, 0) \). If both bidders participate, they form their bidding strategies with common knowledge that the rival’s signals are distributed uniformly over \( [\hat{s}_S, \hat{s}] \). The resulting expected net payoff of a bidder with signal \( s \) is

\[
\Pi_S(b, s, \hat{s}) - C = \frac{\hat{s}_S}{\hat{s}} \int_0^{\hat{s}_S} (v(\hat{s}, x) - v(0, 0)) \frac{1}{\hat{s}_S} dx + \frac{\hat{s} - \hat{s}_S}{\hat{s}} \int_{\hat{s}_S}^{\hat{s}} (v(\hat{s}, x) - b) \frac{1}{\hat{s} - \hat{s}_S} dx - C. \tag{29}
\]

If both bidders participate, taking the first order condition, we obtain the equivalent of (6),

\[
\left( \frac{\partial a_S(s, \hat{s})}{\partial s} \right) (v(s, s) - a(s, \hat{s})) - (s - \hat{s}_S) = 0 \tag{30}
\]

with the boundary condition \( a_S(\hat{s}_S, \hat{s}) = v(\hat{s}_S, \hat{s}) \). The solution is similar to (7):

\[
a_S(s) = \int_{\hat{s}}^{s} v(x, x) \frac{1}{s - \hat{s}_S} dx = \mathbb{E}[v(x, x)|x \in [\hat{s}_S, s]]. \tag{31}
\]

Similarly to bidder-initiated auctions, if both bidders participate, the payoff of a bidder with signal \( \hat{s}_S \) is equal to zero. This logic leads to the equilibrium condition that the bidder with the cut-off signal obtains zero expected net payoff:

\[
\Pi_S(a_S(\hat{s}), \hat{s}, \hat{s}) - C = \int_0^{\hat{s}_S} (v(\hat{s}_S, x) - v(0, 0)) \frac{1}{\hat{s}} dx - C = 0. \tag{32}
\]

Because the expected net payoff is increasing in \( \hat{s}_S \), it uniquely defines equilibrium \( \hat{s}_S(\hat{s}) \). In turn, the cut-off signal \( \hat{s} \) is determined from the indifference condition (17). Example 5 in the appendix illustrates the solution for the case \( v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i}) \).

The seller can and sometimes does commit to reimburse a bidder’s participation costs (e.g., see
Wang (2016) for examples in the M&A transactions). Suppose that the seller can decide on the reimbursement. Conditional on bidder initiation, if the seller chooses not to reimburse, he obtains $v(0,0)$. If he chooses to reimburse, he obtains $R_B(s) - 2C = \mathbb{E} \left[ \max \{ a_I(s, \hat{s}), a_N(s, \hat{s}) \} \right] - 2C$. Reimbursement is optimal if and only if $R_B(s) \geq v(0,0) + 2C$, which is a mild assumption in many settings because participation costs are often low. However, if upon being approached, the seller chooses to reimburse, bidders, again, do not have an incentive to approach, because bidder $I$ with signal $\hat{s}$ expects to always face a rival and obtain no surplus. For the seller to enjoy the benefits of bidder initiation, then, he has to commit to not reimburse participation costs at $t = 0$.

5.4 Discussion of other extensions and assumptions

5.4.1 Investment banks

Our results in the common-value framework rely on the assumption that whether the auction is bidder- or seller-initiated is known by auction participants. While it is in the ex-post interest of the seller to disclose the interest of the initiating bidder to other bidders, such disclosure can be ex-ante suboptimal for all parties combined as it impedes initiation. This inconsistency between ex-ante and ex-post objectives can create a role for an intermediary, such as an investment bank, to alleviate the lack-of-commitment problem. The investment bank can centralize communication among all participating parties, and, because it is a long-run player that interacts with buyers and sellers over time across a range of different services, can incentivize the seller to not disclose the ex-post beneficial but ex-ante harmful, for all parties combined, information in the context of a single transaction.  

5.4.2 First- versus second-price auction

We model contests as first-price auctions. While many auctions are formal first-price auctions, auctions of companies are not. However, modeling them as first-price auctions is a reasonable approximation, because the actual format is somewhere between the first-price and ascending-bid auction (see Hansen (2001) for a description). Specifically, there are multiple rounds of informal bidding, in which bids are contingent on further due diligence and acquisition of financing and can easily be retracted, followed by a single round of formal bidding, which proceeds as a first-

\footnote{Hiding whether the auction is bidder- or seller-initiated gives positive rents to the initiating bidder with the cut-off signal, because the rival believes that, with some probability, the auction is seller-initiated and the initiating bidder's signal is below the cut-off.}
price auction with committing bids. Even when there is more than one round of formal bidding, the number of rounds is low, so the outcomes can be close to those in the first-price auction (e.g., Avery, 1998). In principle, it is not difficult to extend our results to second-price auctions. To avoid triviality, the model would need to have different but positively correlated pre- and post-entry bidder signals – otherwise, the initiating bidder wins the second-price auction with probability one, if she bids her own valuation. For example, bidders in takeover auctions update their pre-entry signals by doing extensive due diligence. The drawback of having an update in signals is that the surplus of the initiating bidder with the cut-off signal remains positive, so the model of second-price auctions would not generate the lack of bidder initiation in the pure common-value framework. However, if the update in signals is small, common-value auctions would still result in considerably fewer bidder-initiated deals than private-value auctions.

5.4.3 Non-stationary dynamics

The stationary solution restriction of Section 4 does not inform on how quickly the initiation game reaches stationarity. If the process is slow, the agents are likely to resolve the game using considerably different strategies. To examine non-stationarity, we specialize to the case of private values (the case of common value still results in no bidder initiation). It is reasonable to assume that at time \( t = 0 \), both bidders start with unrestrictedly distributed signals on \([0, 1]\). In the online appendix, we show that there exists an equilibrium of the initiation game, in which the cut-off signal, \( \hat{s}_t \), is decreasing over time, such that \( \hat{s}_0 = 1 \) and \( \lim_{t \to \infty} \hat{s}_t = \hat{s} \), guaranteeing convergence to the stationary solution. Example 7 in the online appendix shows that for reasonable model parameters (\( r = 0.05 \) and \( \lambda = 0.5 \), so that bidders experience a new shock on average every two years), this convergence is fast: 50% of the distance between \( \hat{s}_0 \) and \( \hat{s} \) is covered in 3.5 years, and 90% of the distance is covered in 9.2 years. Hence, it appears that the focus on the stationary solution is not overly restrictive.

6. Empirical Implications

The model delivers many empirical implications. In particular, they are testable in the context of auctions of companies, because U.S. public targets must file deal backgrounds, which include information on initiation, with the SEC. We split implications into two groups: about bidding in bidder- versus seller-initiated auctions and about initiation.
6.1 Implications about bidding

Consider a private-value setting, in which auctions can be either bidder- or seller-initiated (i.e., $\hat{s} < 1$ and $\mu \in (0, \infty)$). How is bidding by the initiating bidder different from bidding by the non-initiating bidder? And how is bidding in a seller-initiated auction different from bidding in a bidder-initiated auction? The model delivers the following implications:

1. In bidder-initiated deals, the initiating bidder is stronger (has, on average, higher valuations) than the other bidders.

2. In bidder-initiated deals, conditional on the same valuation, the non-initiating bidder bids more aggressively than the initiating bidder: $a_N (\hat{s}, \hat{s}) > a_I (\hat{s}, \hat{s})$.

3. In bidder-initiated deals, unconditionally on the exact valuation, the initiating bidder bids more aggressively and wins more often: $\mathbb{E} [a_I (s, \hat{s}) \mid s \geq \hat{s}] > \mathbb{E} [a_N (s, \hat{s}) \mid s \leq \hat{s}]$.

4. All else equal, bidders in seller-initiated auctions are weaker (have lower valuations) than bidders in bidder-initiated auctions.

5. Conditional on the same valuation, bidders bid less aggressively in seller-initiated deals.

6.2 Implications about initiation

The model links the valuation structure to whether the auction is likely to be seller- or bidder-initiated. In particular, assuming that a takeover battle between strategic (financial) bidders is represented by the private-value (common-value) framework, the model delivers the following implications:

1. Contests among financial bidders are more likely to be seller-initiated than contests among strategic bidders.

2. Contests among financial bidders for a target with the entrenched board and management are often initiated by an activist investor pressuring the board to sell the company.

3. In bidder-initiated acquisitions by financial bidders, the initiating bidder is likely to have a toehold.

4. If due diligence is costly (e.g., if the target is complex) and the target does not reimburse the costs, a bidder-initiated common-value auction is likely to be uncontested.
7. Conclusion

In this paper, we theoretically examine endogenous initiation of a first-price auction by potential buyers and the seller. Our model aims to capture many real-world environments in which initiation of an auction is a strategic choice. Examples include corporate takeovers and intercorporate asset sales, as well as auctions of art. We show that in common-value auctions, such as battles between financial bidders for the target company, bidders are reluctant to approach the seller, because this decision erodes their information rents. In pure common-value auctions, this effect is extreme: no bidder ever approaches, and auctions are initiated by the seller, if at all. By contrast, in private-value auctions, the effect can be opposite: observing that no bidder has approached the seller yet reveals information that rivals are weak, which incentivizes sufficiently strong bidders to approach. Bidder and seller initiation become substitutes, which leads to multiple equilibria with different initiation patterns. We extend the model to include various features of the market for corporate control and show that activist investors and, surprisingly, toeholds held by bidders in the target can alleviate the inefficiencies in common-value auctions and result in more positive-value deals.

Two applied extensions of the paper could be interesting. First, it can be useful to consider bids in securities, such as a bidder’s stock, and the interaction of securities used in a bid and initiation decisions. Second, the asset for sale can be made divisible: for example, bankrupt companies are often sold piecemeal in a liquidation auction. A more general extension is to consider multiple sellers with similar assets for sale and allow bidders to choose which asset to pursue and sellers to choose which bidders to invite to an auction in a dynamic model of matching.
References


Appendix

A Additional proofs

Proof of Lemma 3. Re-write (6) as

\[ \frac{s}{\partial s} a_S (s, \hat{s}) + a_S (s, \hat{s}) = v (s, s) \Rightarrow \frac{d (s a_S (s, \hat{s}))}{ds} = v (s, s). \tag{33} \]

Integrating and applying the initial value condition \( a_S (0, \hat{s}) = v (0, 0) \) yields (7).

Proof of Lemma 5. The equilibrium payoff of bidder 1 with signal \( \hat{s} \) is

\[ \Pi^*_I (\hat{s}, \hat{s}) = \frac{\phi_N (a (\hat{s}), \hat{s})}{\hat{s}} (v (\hat{s}) - a (\hat{s})) = \max_{b \in [a (\hat{s}), \pi (\hat{s})]} \frac{\phi_N (b, \hat{s})}{\hat{s}} (v (\hat{s}) - b) \geq \frac{\phi_N (\bar{a} (\hat{s}), \hat{s})}{\hat{s}} (v (\hat{s}) - \bar{a} (\hat{s})) = v (\hat{s}) - \bar{a} (\hat{s}) = \Pi^*_N (\hat{s}, \hat{s}). \tag{34} \]

Therefore, \( \Pi^*_I (\hat{s}, \hat{s}) \geq \Pi^*_N (\hat{s}, \hat{s}) \). Moreover, if \( \hat{s} < 1 \), then \( a_I (\hat{s}, \hat{s}) = a (\hat{s}) < \bar{a} (\hat{s}) \). Therefore, the inequality in (34) is strict.

Proof of Lemma 7. The equilibrium payoff of a bidder with signal \( \hat{s} \) in a seller-initiated auction is

\[ \Pi^*_S (\hat{s}, \hat{s}) = \Pi_S (a_S (\hat{s}), \hat{s}, \hat{s}) \geq \Pi_S (a (\hat{s}), \hat{s}, \hat{s}) = \frac{\phi_S (a (\hat{s}))}{\hat{s}} (v (\hat{s}) - a (\hat{s})) \geq v^{-1} (a (\hat{s})) (v (\hat{s}) - a (\hat{s})) = \Pi^*_I (\hat{s}, \hat{s}), \tag{35} \]

with the strict inequality if \( \phi_S (a (\hat{s}), \hat{s}) > v^{-1} (a (\hat{s})) \). The last inequality follows from \( a_S (s) \leq v (s) \), which implies \( \phi_S (b) \geq v^{-1} (b) \). Intuitively, no bidder bids above his valuation, so if a bidder bids \( b \), then her signal is at least \( v^{-1} (b) \).

Derivations of expressions for \( \Pi^*_I (\hat{s}), \Pi^*_N (\hat{s}), \) and \( \Pi^*_S (\hat{s}) \) in (20). First, consider \( \Pi^*_S (\hat{s}) \):

\[ \Pi^*_S (\hat{s}) = \frac{1}{\hat{s}} \int_0^{\hat{s}} \Pi^*_S (s, \hat{s}) ds, \text{ where } \Pi^*_S (s, \hat{s}) = \max_b \frac{\phi_S (b, \hat{s})}{\hat{s}} (v (s) - b). \tag{36} \]

By the envelope theorem,

\[ \frac{\partial \Pi^*_S (s, \hat{s})}{\partial s} = \frac{s}{\hat{s}} v' (s) \Rightarrow \Pi^*_S (s, \hat{s}) = \frac{1}{\hat{s}} \int_0^s xv' (x) dx. \tag{37} \]

Plugging into (36) and changing the order of integration,

\[ \Pi^*_S (\hat{s}) = \int_0^{\hat{s}} \frac{s}{\hat{s}} \left( 1 - \frac{s}{\hat{s}} \right) v' (s) ds. \tag{38} \]

Second, consider \( \Pi^*_N (\hat{s}) \). In the range \( s \leq v^{-1} (a (\hat{s})) \), \( \Pi^*_N (s, \hat{s}) = 0 \), because bidder \( N \) never wins.
In the range $s > v^{-1}(a(\hat{s}))$,$$
abla_{\hat{\mathcal{N}}}^{*}(s,\hat{s}) = \max_{b} \frac{\phi_{I}(b, \hat{s}) - \hat{s}}{1 - \hat{s}} (v(s) - b).$$\tag{39}

By the envelope theorem,$$rac{\partial \nabla_{\hat{\mathcal{N}}}^{*}(s,\hat{s})}{\partial s} = \frac{\phi_{I}(a_{N}(s,\hat{s}), \hat{s}) - \hat{s}}{1 - \hat{s}} v'(s) \Rightarrow \nabla_{\hat{\mathcal{N}}}^{*}(s,\hat{s}) = \int_{v^{-1}(a(\hat{s}))}^{s} \frac{\phi_{I}(a_{N}(x,\hat{s}), \hat{s}) - \hat{s}}{1 - \hat{s}} v'(x) dx. \tag{40}

Changing the order of integration,$$
abla_{\hat{\mathcal{N}}}^{*}(\hat{s}) = \frac{1}{\hat{s}} \int_{0}^{\hat{s}} \nabla_{\hat{\mathcal{N}}}^{*}(s,\hat{s}) ds = \frac{1}{\hat{s}} \int_{v^{-1}(a(\hat{s}))}^{\hat{s}} \frac{\phi_{I}(a_{N}(x,\hat{s}), \hat{s}) - \hat{s}}{1 - \hat{s}} v'(x) dx ds$$

$$= \int_{v^{-1}(a(\hat{s}))}^{\hat{s}} \frac{\phi_{I}(a_{N}(s,\hat{s}), \hat{s}) - \hat{s}}{1 - \hat{s}} (1 - \frac{s}{\hat{s}}) v'(s) ds. \tag{41}

Finally, consider $\nabla_{I}^{*}(\hat{s})$. $\nabla_{I}^{*}(s,\hat{s})$ is equal to
\[\nabla_{I}^{*}(s,\hat{s}) = \max_{b} \frac{\phi_{N}(b,\hat{s})}{\hat{s}} (v(s) - b).\tag{42}\]

By the envelope theorem,$$rac{\partial \nabla_{I}^{*}(s,\hat{s})}{\partial s} = \frac{\phi_{N}(a_{I}(s,\hat{s}), \hat{s})}{\hat{s}} v'(s) \Rightarrow \nabla_{I}^{*}(s,\hat{s}) = \nabla_{I}^{*}(\hat{s},\hat{s}) + \int_{\hat{s}}^{s} \frac{\phi_{N}(a_{I}(x,\hat{s}), \hat{s})}{\hat{s}} v'(x) dx. \tag{43}\]

Changing the order of integration,$$
abla_{I}^{*}(\hat{s}) = \frac{1}{1 - \hat{s}} \int_{\hat{s}}^{1} \nabla_{I}^{*}(s,\hat{s}) ds = \nabla_{I}^{*}(\hat{s},\hat{s}) + \frac{1}{1 - \hat{s}} \int_{\hat{s}}^{1} \int_{\hat{s}}^{s} \frac{\phi_{N}(a_{I}(x,\hat{s}), \hat{s})}{\hat{s}} v'(x) dx ds$$

$$= \max_{b} \frac{\phi_{N}(a_{I}(s,\hat{s}), \hat{s})}{\hat{s}} (v(s) - b) + \int_{\hat{s}}^{1} \frac{\phi_{N}(a_{I}(s,\hat{s}), \hat{s})}{\hat{s}} \frac{1 - s}{1 - \hat{s}} v'(s) ds. \tag{44}\]

**Proof of Proposition 1.** First, suppose that $\hat{s} \in (0,1)$, and consider a bidder whose signal just switched to $\hat{s} + \varepsilon$ for an infinitesimal positive $\varepsilon$. By initiating the auction immediately, the bidder gets the expected payoff of $\nabla_{I}^{*}(\hat{s} + \varepsilon, \hat{s}) \equiv \nabla_{I}(a_{I}(\hat{s} + \varepsilon, \hat{s}), \hat{s} + \varepsilon, \hat{s})$, which converges to zero as $\varepsilon \to 0$. Consider a deviation to the following strategy: the bidder waits until either the seller or the rival initiates the auction, or until her signal switches (and initiates the auction then).

The expected payoff from this strategy is
\[
\frac{\lambda (1 - \hat{s}) \nabla_{\hat{\mathcal{N}}}^{*}(\hat{s} + \varepsilon, \hat{s}) + \mu \nabla_{S}^{*}(\hat{s} + \varepsilon, \hat{s}) + \lambda \int_{\hat{s}}^{1} \nabla_{I}^{*}(s,\hat{s})}{r + \lambda (1 - \hat{s}) + \mu + \lambda},
\tag{45}\]

where $\mu dt$ is the probability with which the seller initiates the auction at any short period $(t, t + dt)$ in the stationary equilibrium, $\nabla_{\hat{\mathcal{N}}}^{*}(\hat{s} + \varepsilon, \hat{s}) \equiv \max_{b} \nabla_{\hat{\mathcal{N}}}(b, \hat{s} + \varepsilon, \hat{s}) \geq \nabla_{\hat{\mathcal{N}}}(\hat{s}, \hat{s})$, $\nabla_{S}^{*}(\hat{s} + \varepsilon, \hat{s}) \equiv \max_{b} \nabla_{S}(b, \hat{s} + \varepsilon, \hat{s}) \geq \nabla_{S}^{*}(\hat{s}, \hat{s})$ and $\nabla_{I}^{*}(s,\hat{s}) \equiv \nabla_{I}(a_{I}(s,\hat{s}), \hat{s}, \hat{s})$ for any $s \in [\hat{s}, 1]$. Because
\[ \Pi_N^* (\hat{s}, \hat{s}) \text{ and } \Pi_S^* (\hat{s}, \hat{s}) \text{ are strictly positive for any } \hat{s} > 0, \ \Pi_N^* (\hat{s} + \varepsilon, \hat{s}) \text{ and } \Pi_S^* (\hat{s} + \varepsilon, \hat{s}) \text{ are strictly positive and bounded away from zero. For any } \hat{s} < 1, \int_0^{\hat{s}} \Pi_I^* (s, \hat{s}) \text{ is also strictly positive and bounded away from zero. Therefore, because } r \text{ is finite, the bidder gets a strictly higher expected payoff from this deviation than from the conjectured equilibrium strategy for any } \mu. \]

Second, consider \( \hat{s} = 0 \). Because \( \int_0^1 \Pi_I^* (s, 0) \) is strictly positive and bounded away from zero, the bidder does not benefit from the deviation if and only if \( \mu = \infty \), i.e., if she expects the seller to initiate the auction immediately. However, this situation \( (\hat{s} = 0, \ \mu = \infty) \) is observationally equivalent to \( \hat{s} = 1, \ \mu = \infty \).

**Proof of Proposition 2.** Let \( \hat{s} \) be the signal of a bidder indifferent between approaching the seller and waiting (i.e., \( \hat{s} \) satisfies (17)). It corresponds to a stationary cut-off equilibrium if and only if all bidders with signals \( s > \hat{s} \) find it optimal to approach the seller and all bidders with signals \( s < \hat{s} \) find it optimal to wait:

\[ \Pi_I^* (s, \hat{s}) \geq \frac{\lambda (1 - \hat{s}) \Pi_N^* (s, \hat{s}) + \mu \Pi_S^* (s, \hat{s}) + \lambda X (\hat{s})}{r + \lambda (1 - \hat{s}) + \lambda + \mu} \quad (46) \]

for any \( s > \hat{s} \), and

\[ \Pi_I^* (s, \hat{s}) \leq \frac{\lambda (1 - \hat{s}) \Pi_N^* (s, \hat{s}) + \mu \Pi_S^* (s, \hat{s}) + \lambda X (\hat{s})}{r + \lambda (1 - \hat{s}) + \lambda + \mu} \]

for any \( s < \hat{s} \). First, consider \( s > \hat{s} \). In this range,

\[ \Pi_N^* (s, \hat{s}) = v(s) - a_S (\hat{s}) \text{ for } s > \hat{s} \quad \Rightarrow \quad \frac{\partial \Pi_N^* (s, \hat{s})}{\partial s} = \frac{\partial v}{\partial s} (s) \text{ for } s > \hat{s}; \quad (47) \]

\[ \Pi_S^* (s, \hat{s}) = v(s) - a_S (\hat{s}) \text{ for } s > \hat{s} \quad \Rightarrow \quad \frac{\partial \Pi_S^* (s, \hat{s})}{\partial s} = \frac{\partial v}{\partial s} (s) \text{ for } s > \hat{s}. \quad (48) \]

Thus, the derivative of the right-hand side of (46) in \( s \) is

\[ \frac{\partial \text{RHS}}{\partial s} = \frac{\lambda (1 - \hat{s}) + \mu}{r + \lambda (1 - \hat{s}) + \lambda + \mu} \frac{\partial v}{\partial s} (s). \quad (49) \]

The derivative of the left-hand side of (46) in \( s \) is derived in (43). Whenever \( \text{LHS} (s) = \text{RHS} (s) \), bidders with signals \( s \) in the neighborhood of \( \hat{s} \) find it optimal to approach the seller if and only if

\[ \phi_N (a_I (s, \hat{s}), \hat{s}) > \frac{\lambda (1 - \hat{s}) + \mu}{r + \lambda (1 - \hat{s}) + \lambda + \mu} . \quad (50) \]

The left-hand side of (50) is increasing in \( s \) \( (a_I (s, \hat{s}) \) is increasing in \( s \), so \( \phi_N (a_I (s, \hat{s}), \hat{s}) \) is also increasing in \( s \). Thus, it is sufficient to verify that the inequality holds for \( s \downarrow \hat{s} \). Note that \( a_I (\hat{s}, \hat{s}) = a (\hat{s}) \). Then, \( \phi_N (a (\hat{s}), \hat{s}) = v^{-1} (a (\hat{s})) \). Thus, the equilibrium condition on \( \hat{s} \) and \( \mu \) is

\[ \frac{v^{-1} (a (\hat{s}))}{\hat{s}} > \frac{\lambda (1 - \hat{s}) + \mu}{r + \lambda (1 - \hat{s}) + \lambda + \mu} . \quad (51) \]
Next, consider \( s < \hat{s} \). In this range,

\[
\Pi_I^*(s, \hat{s}) = \max \left\{ \frac{v^{-1}(\hat{q}(\hat{s}))}{\hat{s}} (v(s) - \hat{q}(\hat{s})), 0 \right\} \Rightarrow \frac{\partial \Pi_I^*(s, \hat{s})}{\partial s} = \frac{v^{-1}(\hat{q}(\hat{s}))}{\hat{s}} v'(s) \text{ if } v(s) > \hat{q}(\hat{s}). \tag{52}
\]

Expressions for \( \frac{\partial \Pi_N(s, \hat{s})}{\partial s} \) and \( \frac{\partial \Pi_S(s, \hat{s})}{\partial s} \) for \( s < \hat{s} \) are derived in (40) and (37). Thus, the derivative of the right-hand side of (46) in \( s \) is

\[
\frac{\partial \text{RHS}}{\partial s} = \frac{\lambda (1 - \hat{s}) \phi_I(\alpha_N(s, \hat{s}), \hat{s}) - \hat{s}}{r + \lambda (1 - \hat{s}) + \lambda + \mu} + v'(s). \tag{53}
\]

Whenever \( \text{LHS}(s) = \text{RHS}(s) \), bidders with signals \( s \) in the neighborhood of \( \hat{s} \) find it not optimal to approach the seller if and only if

\[
\frac{v^{-1}(\hat{q}(\hat{s}))}{\hat{s}} > \frac{\lambda (1 - \hat{s}) \phi_I(\alpha_N(s, \hat{s}), \hat{s}) - \hat{s}}{r + \lambda (1 - \hat{s}) + \lambda + \mu}, \tag{54}
\]

The right-hand side of (54) is strictly increasing in \( s \). Thus, it is sufficient to verify that the inequality holds for \( s \uparrow \hat{s} \). Note that \( \alpha_N(\hat{s}, \hat{s}) = \hat{a}(\hat{s}) \). Then, \( \phi_I(\hat{a}(\hat{s}), \hat{s}) = 1 \). Thus, the equilibrium condition on \( \hat{s} \) and \( \mu \), again, simplifies to (51). Thus we have the "if and only if" result. We can re-write (51) as a restriction (18) on \( \mu \), completing the proof of the proposition.

**Proof of Proposition 4.** First, consider the decision of the seller to delay initiation. For \( \hat{s} = 1, \frac{2\lambda(1-\hat{s})}{r+2\lambda(1-\hat{s})} R_E(1) = 0 \), while \( R_S(1) > 0 \). Thus, by Proposition 3, the seller does not benefit from the delay. Second, consider the decision of a bidder to initiate the auction. Note that

\[
\Pi_I^*(1, 1) < \Pi_S^*(1, 1) = \lim_{\mu \to \infty} \frac{\lambda (1 - \hat{s}) \Pi_N^*(s, \hat{s}) + \mu \Pi_S^*(s, \hat{s}) + \lambda X(\hat{s})}{r + \lambda (1 - \hat{s}) + \lambda + \mu} \tag{55}
\]

by Lemma 7, implying that a bidder with signal 1 does not benefit from immediate initiation. Further, from (52)–(54) in Proposition 2 it follows that \( \Pi_I^*(s, 1) < \Pi_S^*(s, 1) \), so no bidder with signal \( s < 1 \) benefits from immediate initiation.

**Proof of Proposition 7.** We compute the implied price of the target, \( P(q) \), conditional on the market-maker receiving the order from either the activist or the liquidity trader. If the activist acquires fraction \( \alpha \) with probability \( q \) upon discovering the target, the Bayes’ rule implies that the probability that the order comes from the activist, conditional on receiving the order, is \( \frac{\lambda Aq}{\lambda Aq + \lambda L} \). Therefore, the price equals

\[
P(q) = \frac{\lambda Aq}{\lambda Aq + \lambda L} R_S(1) + \frac{\lambda L}{\lambda Aq + \lambda L} \int_0^\infty e^{-rt} R_S(1) q \lambda A e^{-q\lambda A t} dt = R_S(1) \frac{\lambda Aq}{\lambda Aq + \lambda L} \left( 1 + \frac{\lambda L}{r + \lambda Aq} \right). \tag{56}
\]

Hence, the activist’s expected payoff from acquiring fraction \( \alpha \) and undertaking the campaign is \( \alpha R_S(1) - \alpha P(q) - A \), yielding (22). If (22) is above zero, the activist is better off acquiring the block; if (22) is below zero, it is better not acquiring the block; and if (22) is equal to zero, it is indifferent. Because (22) is strictly decreasing in \( q \geq 0 \), the statement of Proposition 7 follows,
with \( q \), in the case of \( A \in (A, \overline{A}) \), given by the quadratic equation

\[
\frac{r \lambda L \alpha R_S(1)}{A} = (q \lambda_A + \lambda_L) (r + q \lambda_A) \Rightarrow q_{1,2} = \frac{-(r + \lambda_L) \pm \sqrt{(r - \lambda_L)^2 + 4 \lambda_L r \alpha R_S(1)}}{2}.
\]

Because \( q \in [0,1] \) by definition and the lower root is negative, the upper root is the relevant one.

**Model with toeholds.** We compute the implied price of the target conditional on the market-maker receiving an offer from either a bidder or the liquidity trader. Suppose that a bidder that considers approaching the target acquires fraction \( \alpha \) with probability \( q \). Then, the Bayes’ rule implies that the probability that the order comes from one of the two bidders, conditional on receiving the order, is \( \frac{2 \lambda (1 - \hat{s})^q}{2 \lambda (1 - \hat{s})^q + \lambda L} \). Therefore, the price equals:

\[
P(q) = \frac{2q \lambda (1 - \hat{s})}{2q \lambda (1 - \hat{s}) + \lambda L} R_B (\hat{s}) + \frac{\lambda L}{2q \lambda (1 - \hat{s}) + \lambda L} R_S (\hat{s}) + \mu R_S (\hat{s}) e^{-(2q \lambda (1 - \hat{s}) + \mu) t} dt
\]

for \( q = 1 \), the expression simplifies to

\[
P = \frac{2 \lambda (1 - \hat{s})}{2 \lambda (1 - \hat{s}) + \lambda L} \left( 1 + \frac{\lambda L}{r + 2 \lambda (1 - \hat{s}) + \mu} \right) R_B (\hat{s}) + \frac{\mu \lambda L}{(2 \lambda (1 - \hat{s}) + \lambda L)(r + 2 \lambda (1 - \hat{s}) + \mu)} R_S (\hat{s}).
\]

If, in addition, the focus is on equilibria, in which the seller does not initiate the auction (\( \mu = 0 \)), the expression further simplifies to (23).

Next, we extend the analysis of Section 5.2 to characterize equilibria, in which both bidders and the seller initiate the auction with positive probabilities. If \( \mu > 0 \), the payoff to bidder \( I \) with signal \( \hat{s} \) from the auction, net of the cost of acquiring toehold, is

\[
\Pi_I (\hat{s}, \hat{s}) = \alpha \left( \int_0^\infty a_N (x, \hat{s}) \frac{1}{\hat{s}} dx - \frac{2 \lambda (1 - \hat{s}) R_B (\hat{s})}{2 \lambda (1 - \hat{s}) + \lambda L} \left( 1 + \frac{\lambda L}{r + 2 \lambda (1 - \hat{s}) + \mu} \right) - \frac{\mu \lambda L R_S (\hat{s})}{(2 \lambda (1 - \hat{s}) + \lambda L)(r + 2 \lambda (1 - \hat{s}) + \mu)} \right).
\]

For an equilibrium, first, a bidder with signal \( \hat{s} \) must be indifferent between approaching the seller and waiting. The cut-off signal is determined from the indifference condition (17). Second, the seller must be indifferent between initiating the auction and waiting: \( \frac{2 \lambda (1 - \hat{s})}{r + 2 \lambda (1 - \hat{s})} R_B (\hat{s}) = R_S (\hat{s}) \).

**B Examples**

**Example 1:** auction stage, the common-value framework. Suppose that \( v (s_i, s_{-i}) = \frac{1}{2} (s_i + s_{-i}) \). To find the equilibrium, let \( \phi_j (b, \hat{s}) = \alpha_j + \beta_j b, j \in \{I, N\} \). Then, the first two
boundary conditions in (4) become:

\[ 1 = \alpha_I + \beta_I \hat{a}(\hat{s}), \quad \hat{s} = \alpha_N + \beta_N \hat{a}(\hat{s}) \quad \Rightarrow \quad 1 + \hat{s} = \alpha_I + \alpha_N + (\beta_I + \beta_N) \hat{a}(\hat{s}). \quad (62) \]

The second two boundary conditions become:

\[ \hat{s} = \alpha_I + \beta_I \frac{\hat{s}}{2}, \quad 0 = \alpha_N + \beta_N \frac{\hat{s}}{2} \quad \Rightarrow \quad \hat{s} = \alpha_I + \alpha_N + (\beta_I + \beta_N) \frac{\hat{s}}{2}. \quad (63) \]

The difference between (62) and (63) yields \( \beta_I + \beta_N = \frac{1}{(\hat{a}(\hat{s}) - \hat{s}/2)}. \) Next, plugging \( \phi_j(b, \hat{s}) \) into differential equations (3) yields:

\[ \beta_j = \frac{\alpha_j + \beta_j b - \hat{s}_j}{\frac{1}{2}(\alpha_I + \alpha_N + (\beta_I + \beta_N) b) - b}. \quad (64) \]

Adding the two equations up at \( b = \hat{a}(\hat{s}) \) results in \( \beta_I + \beta_N = \frac{1}{(\hat{a}(\hat{s}) - \hat{s}/2)}. \) Combining the two equations for \( \beta_I + \beta_N \) yields \( \hat{a}(\hat{s}) = \frac{1+2\hat{s}}{4} \) and the range of bids \( b \in \left[ \frac{\hat{s}}{2}, \frac{1+2\hat{s}}{4} \right] \) for both bidders. With \( \hat{a}(\hat{s}) \) known, the coefficients in \( \phi_j(b, \hat{s}) \) can be found from boundary conditions. The resulting inverses of bidding strategies are:

\[ \phi_I(b, \hat{s}) = \hat{s}(2\hat{s} - 1) + 4(1 - \hat{s})b, \quad \phi_N(b, \hat{s}) = -2\hat{s}^2 + 4\hat{s}b. \quad (65) \]

The bidding strategies given signals are, in turn, inverses of \( \phi_j(b, \hat{s}) \). They are linear in own signals:

\[ a_I(s, \hat{s}) = \frac{s + \hat{s}(1 - 2\hat{s})}{4(1 - \hat{s})}, \quad a_N(s, \hat{s}) = \frac{s + 2\hat{s}^2}{4\hat{s}}. \quad (66) \]

**Example 2: auction stage, the private-value framework.** Suppose that \( v(s_i) = s_i \). The solution to the system of differential equations (10) subject to boundary conditions (12) and the minimum serious bid \( \frac{\hat{s}}{2} \) obtained from (11), derived by Kaplan and Zamir (2012), is

\[ \phi_I(b, \hat{s}) = \hat{s} + \frac{\hat{s}^2}{(\hat{s} - 2b) c_I e^{\frac{s}{4\hat{s}} - 4b}}, \phi_N(b, \hat{s}) = \frac{\hat{s}^2}{(\hat{s} - 2b) c_N e^{\frac{s}{4\hat{s}} + 4(\hat{s} - b)}}, \quad (67) \]

where constants \( c_N \) and \( c_I \) are determined from boundary conditions \( \phi_I(\hat{s} - \frac{s^2}{T}) = 1 \) and \( \phi_N(\hat{s} - \frac{s^2}{T}) = \hat{s} \). The range of bids is \( b \in \left[ \frac{\hat{s}}{2}, \hat{s} - \frac{s^2}{4T} \right] \) for both bidders.

**Example 3: initiation stage, the private-value framework.** Suppose that \( v(s_i) = s_i \). The equilibrium in the seller-initiated auction is \( a_S(s) = \frac{s}{2} \) and the range of bids is \( [0, \frac{s}{2}] \). The implied expected revenues of the seller are

\[ R_S(\hat{s}) = 2 \int_0^{\hat{s}} s \left( 1 - \frac{s}{\hat{s}} \right) \frac{1}{\hat{s}} ds = \frac{\hat{s}^3}{3}. \quad (68) \]
Consider the initiation game. Restriction \( \mu < \hat{\mu}(\hat{s}) \) in this case simplifies to\(^20\)

\[
\mu < r + \lambda \hat{s}.
\]

The indifference condition for \( \hat{s} \), (20), simplifies to\(^21\)

\[
\begin{align*}
& r \left( \frac{\hat{s}}{4} + D(\hat{s}, \mu) \Pi'_T(\hat{s}) \right) + \lambda (1 - \hat{s}) \left( \frac{\hat{s}(1 - \hat{s})}{4} + D(\hat{s}, \mu) \left( \Pi'_T(\hat{s}) - \Pi'_N(\hat{s}) \right) \right) \\
= & \ \mu \left( \frac{\hat{s}}{4} + D(\hat{s}, \mu) \left( \frac{\hat{s}}{6} - \Pi'_T(\hat{s}) \right) \right) + \lambda \left( \Pi'_T(\hat{s}) - \frac{\hat{s}}{4} \right). 
\end{align*}
\]

Finally, the seller’s decision problem reduces to the comparison of \( R_B(\hat{s}) \) with \( \left(1 + \frac{r}{2\lambda(1-\hat{s})}\right) \hat{s}^3/3 \).

**Example 4: the common-value framework with toeholds.** Suppose that \( v(s_i, s_{-i}) = \frac{1}{2} (s_i + s_{-i}) \). The general solution to the system of differential equations (26) is given by

\[
\phi_I(b, \hat{s}) = \hat{s} + c_1 \left((1 - \alpha)\phi_N(b, \hat{s})\right)^{\frac{1}{1-\alpha}},
\]

\[
b = \frac{\phi_N(b, \hat{s})}{2(2 - \alpha)} + c_2 \phi_N(b, \hat{s})^{-\frac{1}{1-\alpha}} + \frac{\hat{s}}{2} + \frac{c_1}{4} \left((1 - \alpha)\phi_N(b, \hat{s})\right)^{\frac{1}{1-\alpha}}.
\]

First, use boundary conditions

\[
\hat{s} = \phi_I(v(\hat{s}, 0), \hat{s})
\]

and

\[
0 = \phi_N(v(\hat{s}, 0), \hat{s})
\]

to show that (72) can only be satisfied with equality if \( c_2 = 0 \). Second, plug in \( 1 = \phi_I(\alpha(\hat{s}), \hat{s}) \) and \( \hat{s} = \phi_N(\alpha(\hat{s}), \hat{s}) \) in (71) to obtain

\[
c_1 = \frac{1 - \hat{s}}{(1-\alpha)\hat{s}^{1-\alpha}}.
\]

Plugging \( \hat{s} = \phi_N(\alpha(\hat{s}), \hat{s}) \) and expressions for \( c_1 \) and \( c_2 \) into (72), we obtain \( \alpha(\hat{s}) \):

\[
\alpha(\hat{s}) = \frac{1 + \hat{s}}{4} + \frac{\hat{s}}{2(2 - \alpha)}. \tag{73}
\]

The upper boundary on bids is increasing in \( \alpha \), consistent with the intuition that toeholds result in a more aggressive bidding. Additionally, (72) becomes

\[
b = \frac{\phi_N(b, \hat{s})}{2(2 - \alpha)} + \frac{\hat{s}}{2} + \frac{1 - \hat{s}}{4} \left( \frac{\phi_N(b, \hat{s})}{\hat{s}} \right)^{\frac{1}{1-\alpha}}. \tag{74}
\]

Equivalently, the equilibrium bid of bidder \( N \) with signal \( s \) is

\[
a_N(s, \hat{s}) = \frac{s}{2(2 - \alpha)} + \frac{\hat{s}}{2} + \frac{1 - \hat{s}}{4} \left( \frac{s}{\hat{s}} \right)^{\frac{1}{1-\alpha}}. \tag{75}
\]

Plugging the expression for \( c_1 \) into (71), we obtain

\[
\phi_I(b, \hat{s}) = \hat{s} + (1 - \hat{s}) \left( \frac{\phi_N(b, \hat{s})}{\hat{s}} \right)^{\frac{1}{1-\alpha}}. \tag{76}
\]

\(^{20}\)This can be seen from \( \hat{\mu}(\hat{s}) = \frac{\alpha(\hat{s})}{\hat{s}} (r + \lambda) - \lambda (1 - \hat{s}) = r + \lambda \hat{s}. \)

\(^{21}\)In the alternative specification, where upon experiencing a shock, an existing bidder exits the game, obtains an exogenous payoff \( X > 0 \), and is replaced by a new bidder, the indifference condition for the example further reduces to a cubic equation in \( \hat{s} \) with a closed-form solution.
Substituting the non-linear term in (76) using (74), we obtain
\[
\phi_I(b, \hat{s}) = 4b - \hat{s} - \frac{2\phi_N(b, \hat{s})}{2 - \alpha},
\] (77)

Equivalently, the equilibrium bid of bidder \( I \) with signal \( s \) is
\[
a_I(s, \hat{s}) = \frac{s + \hat{s}}{4} + \frac{\phi_N(a_I(s, \hat{s}), \hat{s})}{2(2 - \alpha)},
\] (78)

where \( \phi_N(a_I(s, \hat{s}), \hat{s}) \) is the signal of bidder \( N \) corresponding to bid \( a_I(s, \hat{s}) \) made by bidder \( I \) with signal \( s \). For the case \( \alpha = 0 \) (no toeholds), \( \phi_N(a_I(s, \hat{s}), \hat{s}) = 4\hat{s}a_I(s, \hat{s}) - 2\hat{s}^2 \), and we have the same solution as in Example 1. For the case \( \alpha > 0 \), a numerical inversion is needed.

To solve for \( \hat{s} \) using the indifference condition (17), we compute the equilibrium payoff from the auction of bidder \( I \) with signal \( \hat{s} \):
\[
\Pi_I^s(\hat{s}, \hat{s}) = \alpha \int_0^{\hat{s}} a_N(s, \hat{s}) \frac{1}{\hat{s}} ds = \alpha \left( \frac{1 - \alpha (1 - \hat{s})}{4(2 - \alpha)} + \frac{\hat{s}}{2} \right).
\] (79)

Equilibrium payoffs of bidders \( N \) and both bidders in a seller-initiated auction with signals \( \hat{s} \) are
\[
\Pi_N^s(\hat{s}, \hat{s}) = \int_{\hat{s}}^1 \left( \frac{\hat{s} + x}{2} - a_s(\hat{s}) \right) \frac{1}{1 - \hat{s}} dx = \frac{\hat{s}}{2(2 - \alpha)},
\] (80)
\[
\Pi_s^s(\hat{s}, \hat{s}) = \int_0^{\hat{s}} \left( \frac{\hat{s} + x}{2} - a_s(\hat{s}) \right) \frac{1}{\hat{s}} dx = \frac{\hat{s}}{4}.
\] (81)

Next, we need to calculate the equilibrium price of fraction \( \alpha \) of the seller. Let \( H_B(b, \hat{s}) \) denote the c.d.f. of the winning bid \( b \) in the bidder-initiated auction:
\[
H_B(b, \hat{s}) = \Pr(a_I(s_1, \hat{s}) \leq b, a_N(s_2, \hat{s}) \leq b) = \Pr(s_1 \leq \phi_I(b, \hat{s}), s_2 \leq \phi_N(b, \hat{s}))
\]
\[
= \frac{\phi_I(b, \hat{s}) - \hat{s}}{1 - \hat{s}} \phi_N(b, \hat{s}) = \frac{4b - 2\hat{s} - \frac{2\phi_N(b, \hat{s})}{2 - \alpha}}{1 - \hat{s}} \phi_N(b, \hat{s}),
\] (82)

so that the corresponding p.d.f. is
\[
h_B(b, \hat{s}) = \frac{4 - \frac{2}{2 - \alpha} \phi'_{N,1}(b, \hat{s}) \phi_N(b, \hat{s})}{1 - \hat{s}} + \frac{4b - 2\hat{s} - \frac{2}{2 - \alpha} \phi_N(b, \hat{s}) \phi'_{N,1}(b, \hat{s})}{1 - \hat{s}}
\]
\[
= \frac{4\phi_N(b, \hat{s}) + 4b\phi'_{N,1}(b, \hat{s}) - 2\hat{s}\phi'_{N,1}(b, \hat{s}) - \frac{4}{2 - \alpha} \phi_N(b, \hat{s}) \phi'_{N,1}(b, \hat{s})}{\hat{s}(1 - \hat{s})}.
\] (83)

Here, \( \phi_N(b, \hat{s}) \) is the numerical solution of (74). Applying the implicit function theorem to (74),
\[
\phi'_{N,1}(b, \hat{s}) = \frac{1}{\frac{1}{2(2 - \alpha)} + \frac{1}{4} \frac{1}{2 - \alpha} \phi_N(b, \hat{s})^{1 - \alpha} - \frac{1}{2 - \alpha} - \frac{1}{2 - \alpha}}.
\] (84)

P.d.f. \( h_B(b, \hat{s}) \) can be calculated numerically. Finally, using (27), the necessary condition for a
bidder to acquire a toehold and initiate the auction when the seller does not initiate auctions \((\mu = 0)\) is
\[
\alpha \left( \frac{1 - \alpha (1 - \hat{s})}{4 (2 - \alpha)} + \frac{1}{2} \hat{s} \right) - \frac{2\lambda (1 - \hat{s})}{2\lambda (1 - \hat{s}) + \lambda_L} \left( 1 + \frac{\lambda_L}{r + 2\lambda (1 - \hat{s})} \right) \int_{\hat{a}(\hat{s})}^{\hat{a}(\hat{s} + \hat{s})} h(b, \hat{s}) db. \tag{85}
\]

Example 5: the common-value framework with preemption. Suppose that \(v(s_i, s_{-i}) = \frac{1}{2}(s_i + s_{-i})\). Using (32), the cut-off on the signal of the participating bidder in a seller-initiated auction is
\[
\hat{s}_S(\hat{s}) = \sqrt{\frac{4}{3}} \hat{s}C. \tag{86}
\]

The equilibrium in the seller-initiated auction with two participating bidders is
\[
a_S(\hat{s}) = \mathbb{E}[x|x \in [\hat{s}_S, \hat{s}]] - \frac{\hat{s}_S + \hat{s}}{2}. \tag{87}
\]
while \(\Pi_N^*(\hat{s}, \hat{s}) = 0\). The equilibrium net payoff of a bidder in the seller-initiated auction is
\[
\Pi^*(\hat{s}, \hat{s}) = \frac{1}{\hat{s}} \left( \int_{0}^{\hat{s}} \frac{\hat{s} + x}{2} dx + \int_{\hat{s}}^{\hat{s}_S} \frac{x - \hat{s}_S}{2} dx \right) - C = \frac{1}{4} \hat{s}^2 - \frac{1}{3} C, \tag{88}
\]
where we use (86), the expression for \(a_S(\hat{s})\), and the equilibrium condition \(\phi_S(a(\hat{s}), \hat{s}) = \hat{s}\). Note that both \(\Pi_I^*(\hat{s}, \hat{s})\) and \(\Pi_N^*(\hat{s}, \hat{s})\) are positive only if \(C < \frac{3}{4} \hat{s}\). This is also the condition for \(\hat{s}_S < \hat{s}\).

To complete the example, we show the existence of the equilibrium with bidder initiation, \(\hat{s} < 1\). If \(\hat{s} = 1\), the indifference condition (17) becomes
\[
(r + \mu + \lambda) \left( \frac{3}{4} - C \right) = \mu \left( \frac{1}{4} - \frac{1}{3} C \right) + \lambda X(1), \tag{89}
\]
where \(X(1) = \frac{1}{1 - \lambda} \frac{\int_{0}^{1} \Pi_N^*(s, 1) ds}{\int_{0}^{1} \Pi_S^*(s, 1) ds}\). If \(\mu\) is sufficiently low and \(C < \frac{3}{4}\), then the left-hand side of (89) exceeds the right-hand side. If, on the other hand, \(\hat{s} > \frac{1}{2} C\), the indifference condition becomes
\[
0 = \lambda X(\hat{s}), \tag{90}
\]
and the right-hand side of (89) exceeds the left-hand side. By continuity, there exists a solution \(\hat{s}\), provided that \(\mu\) is sufficiently low.

C For online publication: Complete dynamics with private values

We focus on the case of private values. Assume that at date \(t = 0\), each potential bidder randomly draws a private signal from the uniform distribution \([0, 1]\). Thus, the distribution of initial and subsequent shocks experienced by bidders is the same. However, to the extent that there exists a stationary cut-off signal \(\hat{s} < 1\), to which the game eventually converges, the game becomes non-stationary in the meantime. To simplify the analysis, we consider non-stationary equilibria,
in which the seller does not initiate the auction \((\mu = 0)\). We conjecture and later confirm that the cut-off \(\hat{s}_t\) is a decreasing function of time: \(\hat{s}_t' < 0\).

In a conjectured non-stationary equilibrium, at any time a bidder can initiate the auction for two reasons. First, she experiences a shock \(s \geq \hat{s}_t\). Second, over time the decreasing cut-off \(\hat{s}_t\) reaches her current signal. Bidder \(N\) cannot distinguish between these two events.

### C.1 Equilibria in bidder- and seller-initiated auctions

First, we solve for the equilibrium at the auction stage. For the remainder of this subsection, to make the notation simpler we omit the subscript for time in \(\hat{s}_t\).

#### C.1.1 A bidder-initiated auction

Because cut-off \(\hat{s}\), by conjecture, is decreasing over time, the probability over a short interval of time \(dt\) that bidder \(I\) received a new signal at or above \(\hat{s}\) is \(\lambda dt(1 - \hat{s})\), while the probability that bidder \(I\)’s old signal was reached by decreasing \(\hat{s}\) is \(-\hat{s}/dt\). The Bayes’ rule implies that conditional on initiation, bidder \(N\) believes the first event occurs with probability \(p = \frac{-\hat{s}/\lambda(1 - \hat{s})}{\hat{s}/\lambda} = 1 - \frac{\hat{s}}{\lambda}\), while the second event occurs with probability \(1 - p\). We conjecture that the equilibrium takes the following form. The set of “serious” bids is \([a(\hat{s}, p), \bar{a}(\hat{s}, p)]\). Bidder \(N\) with signal \(s\) bids \(a_N(s, \hat{s}, p) \in [a(\hat{s}, p), \bar{a}(\hat{s}, p)]\); \(s_i \geq a(s, p)\). Bidder \(I\) with signal \(\hat{s}\) plays the mixed strategy of bidding over interval \([a(\hat{s}, p), \bar{a}(\hat{s}, p)]\) for some \(\bar{a}(\hat{s}, p) \in [a(\hat{s}, p), \bar{a}(\hat{s}, p)]\). Bidder \(I\) with signal \(s_i > \hat{s}\) bids \(a_I(s, \hat{s}, p) \in [\bar{a}(\hat{s}, p), \bar{a}(\hat{s}, p)]\). Note that \(F_I(\bar{a}(\hat{s}, p)) = p\).

The expected payoff of bidder \(N\) with signal \(s\) and bid \(b\) is

\[
\Pi_N(b, s, \hat{s}, p) = E[v(s) - b|x = \hat{s} \text{ or } x \in [\hat{s}, 1]] = (v(s) - b) F_I(b, \hat{s}, p). \tag{91}
\]

Intuitively, bidder \(N\)’s bid exceeds the bid of her rival with probability \(F_I(b, \hat{s}, p)\). In equilibrium, \(b = a_N(s, \hat{s}, p)\), implying \(s = \phi_N(b, \hat{s}, p)\) such that \(\phi_N(b, \hat{s}, p)/\hat{s} = F_N(b, \hat{s}, p)\). Therefore, taking the first-order condition of (91) and using the equilibrium condition, for any \(b \in (a(\hat{s}, p), \bar{a}(\hat{s}, p))\),

\[
\frac{\partial F_I(b, \hat{s}, p)}{\partial b} (v(F_N(b, \hat{s}, p), \hat{s}) - b) = F_I(b, \hat{s}, p). \tag{92}
\]

Next, consider bidder \(I\) with signal \(\hat{s}\). Randomization among bids \(b \in [a(\hat{s}, p), \bar{a}(\hat{s}, p)]\) requires that for any such \(b\) and some constant \(C\), her expected payoff is

\[
\Pi_N(b, \hat{s}, \hat{s}, p) = E[v(\hat{s}) - b|x \in [0, \hat{s}]] = (v(\hat{s}) - b) F_N(b, \hat{s}, p) = C. \tag{93}
\]

If this condition does not hold, bidder \(I\) would deviate to the most profitable set of bids.

Finally, the expected payoff of bidder \(N\) with signal \(s > \hat{s}\) and bid \(b\) is

\[
\Pi_N(b, s, \hat{s}, p) = E[v(s) - b|x \in [0, \hat{s}]] = (v(s) - b) F_N(b, \hat{s}, p). \tag{94}
\]

In equilibrium, \(b = a_I(s, \hat{s}, p)\) for \(s > \hat{s}\), implying \(s = \phi_I(b, \hat{s}, p)\) such that \(p + (1 - p) \frac{\phi_I(b, \hat{s}, p) - \hat{s}}{1 - \hat{s}} = F_I(b, \hat{s}, p)\). Intuitively, the probability for bidder \(N\) to win with bid \(b \geq \bar{a}(\hat{s}, p)\) against bidder \(I\) is \(p \Pr[s < \phi_I(b, \hat{s}, p)|s = \hat{s}] + (1 - p) \Pr[s < \phi_I(b, \hat{s}, p)|s \in [\hat{s}, 1]] = p + (1 - p) \frac{\phi_I(b, \hat{s}, p) - \hat{s}}{1 - \hat{s}}\).
Therefore, taking the first-order condition of (94) and using the equilibrium condition, for any \( b \in [\hat{a}(\hat{s}, p), \bar{a}(\hat{s}, p)] \),

\[
\frac{\partial F_N(b, \hat{s}, p)}{\partial b} \left( v \left( \hat{s} + \frac{F_I(b, \hat{s}, p) - p}{1 - p} (1 - \hat{s}) \right) - b \right) = F_N(b, \hat{s}, p). \tag{95}
\]

The system of three equations, (92), (93), and (95), is solved subject to boundary conditions

\[
1 = F_j(\overline{a}(\hat{s}, p), \hat{s}, p), \quad 0 = F_I(\underline{a}(\hat{s}, p), \hat{s}, p), \quad \frac{v^{-1}(\underline{a}(\hat{s}, p))}{\hat{s}} = F_N(\overline{a}(\hat{s}, p), \hat{s}, p) \tag{96}
\]

for \( j \in \{I, N\} \). These conditions are similar to (12) but also account for randomization by bidder \( I \) with signal \( \hat{s} \). From the third boundary condition and (93), for any \( b \in [\underline{a}(\hat{s}, p), \overline{a}(\hat{s}, p)] \),

\[
F_N(b, \hat{s}, p) = \frac{v^{-1}(\underline{a}(\hat{s}, p))}{\hat{s}} \frac{v(\hat{s}) - \underline{a}(\hat{s}, p)}{v(\hat{s}) - b}. \tag{97}
\]

The minimum bid \( \underline{a}(\hat{s}, p) \) must be optimal for bidder \( I \) with signal \( \hat{s} \). Therefore,

\[
\frac{v^{-1}(\underline{a}(\hat{s}, p))}{\hat{s}} (v(\hat{s}) - \underline{a}(\hat{s}, p)) \geq F_N(b, \hat{s}, p) (v(\hat{s}) - b), \forall b. \tag{98}
\]

In equilibrium, bidder \( N \) never bids above her valuation: \( a_N(s, \hat{s}, p) \leq v(s) \). Therefore, \( \phi_N(b, \hat{s}, p) \geq v^{-1}(b) \), which implies \( F_N(b) \geq \frac{v^{-1}(b)}{\hat{s}} \). Hence,

\[
\frac{v^{-1}(\underline{a}(\hat{s}, p))}{\hat{s}} (v(\hat{s}) - \underline{a}(\hat{s}, p)) \geq \frac{v^{-1}(b)}{\hat{s}} (v(\hat{s}) - b), \forall b. \tag{99}
\]

Therefore,

\[
\underline{a}(\hat{s}, p) = \arg \max_b \frac{v^{-1}(b)}{\hat{s}} (v(\hat{s}) - b) \quad \Rightarrow \quad \text{F.O.C.:} \quad \frac{v'(\hat{s}) - \underline{a}(\hat{s}, p)}{v'(v^{-1}(\underline{a}(\hat{s}, p)))} = v^{-1}(\underline{a}(\hat{s}, p)). \tag{100}
\]

Note that in equilibrium, \( \underline{a}(\hat{s}, p) \equiv \underline{a}(\hat{s}) \). The following lemma summarizes the equilibrium in the bidder-initiated first-price auction in the non-stationary setting:

**Lemma 8 (equilibrium in the bidder-initiated PV auction, the non-stationary setting).** The equilibrium is unique (up to the non-serious bids of types \( s < v^{-1}(\underline{a}(\hat{s}, p)) \) of non-initiating bidders). The equilibrium probabilities of the initiating and non-initiating bidders to win with bid \( b \), \( F_j(b, \hat{s}, p), \ j \in \{I, N\} \), satisfy (92), (93), and (95), with boundary conditions (96) and the lowest serious bid given by (100).

Example 6 derived below provides the closed form solution for the case \( v(s_i) = s_i \). Figure 4 illustrates the equilibrium bids for the case \( \hat{s} = 0.5 \) and \( p = 0.5 \).

**C.1.2 A seller-initiated auction.** If the auction is initiated by the seller, all parties believe that each bidder’s signal is distributed uniformly over \([0, \hat{s}]\). As a result, auction outcomes are identical to those derived in Lemma 6. The following lemma summarizes the equilibrium:
Lemma 9 (equilibrium in the seller-initiated PV auction, the non-stationary setting). The equilibrium is unique. The symmetric equilibrium bidding strategies of the non-initiating bidders, \( a_S(s) \), are increasing functions that are independent of \( s \) and \( p \) and solve (14) with boundary condition \( a_S(0) = v(0) \). Specifically, \( a_S(s) \) is given by (15).

C.2 The initiation game

We solve for symmetric Markov Perfect Bayesian equilibria, in the seller never initiates the auction \((\mu = 0)\). We also derive the conditions, for which this strategy of the seller is optimal.

Suppose that a bidder believes that the rival approaches the seller either because it receives a new signal in excess of \( \hat{s}_t \), which occurs with probability \( \lambda (1 - \hat{s}_t)dt \), or her old signal is reached by decreasing \( \hat{s}_t \), which occurs with probability \(-\hat{s}_t' (t) dt/\hat{s}_t \). Denote the expected continuation value of this bidder by \( V_B(s, t, 0) \equiv V_B(s, \hat{s}_t, p_t, \mu = 0) \). This value satisfies

\[
V(s, t, 0) = \max \left\{ \Pi^*_I(s, \hat{s}_t, p_t), \frac{V_B'(s, t, 0) + (\lambda (1 - \hat{s}_t) - \hat{s}_t'/\hat{s}_t) \Pi^*_N(s, \hat{s}_t, p_t) + \lambda X(t)}{r + \lambda (1 - \hat{s}_t) - \hat{s}_t'/\hat{s}_t + \lambda} \right\}
\]

(101)

where \( X(t) \) is the bidder’s expected continuation value when she experiences a shock: \( X(t) = \int_0^t V_B(s', t, 0) ds' \). The intuition behind (101) is as follows. The continuation value is of the expected payoff from immediate initiation, which yields \( \Pi^*_I(s, \hat{s}_t, p_t) \), and waiting, which yields the second term of (101). The change in the continuation value with time is captured by \( V_B'(s, t, 0) \).

Three independent events can occur if the bidder chooses to wait. First, with intensity \( \lambda (1 - \hat{s}_t) \), a rival with signal above \( \hat{s} \) appears and approaches the seller, and the bidder obtains \( \Pi^*_N(s, \hat{s}_t, p_t) \). Second, with intensity \(-\hat{s}_t'/\hat{s}_t = \frac{p_t}{1 - p_t} \lambda (1 - \hat{s}_t) \), the rival’s signal is reached by decreasing \( \hat{s}_t \), so she approaches the seller, and the bidder obtains \( \Pi^*_N(s, \hat{s}_t, p_t) \). Third, with intensity \( \lambda \), the bidder experiences a shock and obtains \( X(t) \).

By continuity of \( \Pi^*_I(\cdot) \) and \( \Pi^*_N(\cdot) \) in \( s \), the cut-off type must satisfy

\[
P^*_I(\hat{s}_t, \hat{s}_t, p_t) = \frac{V_B'(\hat{s}_t, t, 0) + \lambda \frac{1 - \hat{s}_t}{1 - p_t} \Pi^*_N(\hat{s}_t, \hat{s}_t, p_t) + \lambda X(t)}{r + \lambda \frac{1 - \hat{s}_t}{1 - p_t} + \lambda} \Rightarrow (102)
\]

(102)

In addition, the smooth-pasting condition at \( \hat{s}_t \) must be satisfied:

\[
V_B'(s, t, 0) = \Pi^*_I(\hat{s}_t, \hat{s}_t, p_t) \hat{s}_t' + \Pi^*_N(\hat{s}_t, \hat{s}_t, p_t) p_t'.
\]

(103)

Lemma 10 (Smooth-pasting condition for the value function, the non-stationary setting). The smooth-pasting condition for a bidder with signal \( \hat{s}_t \) is

\[
V_B'(s, t, 0) = -\frac{p_t}{1 - p_t} \lambda \hat{s}_t (1 - \hat{s}_t) \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial \hat{s}_t} (v(\hat{s}_t) - \hat{a}(\hat{s}_t, p_t)),
\]

(104)
where \( F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) = \frac{v^{-1}(a(\hat{s}_t, p_t))}{\hat{s}_t} \left( \frac{v(\hat{s}_t) - a(\hat{s}_t, p_t)}{v(\hat{s}_t) - \hat{a}(\hat{s}_t, p_t)} \right) \).

**Proof.** First, take a partial derivative of \( \Pi^*_I(s, \hat{s}_t, p_t) = \Pi_I(a_I(s, \hat{s}_t, p_t), s, \hat{s}_t, p_t) \) with respect to \( \hat{s}_t \) and compute it at point \( s \downarrow \hat{s}_t \), using that for bidder \( I \), \( \lim_{s \downarrow \hat{s}_t} a_I(s, \hat{s}_t, p_t) = \hat{a}(\hat{s}_t, p_t) \):

\[
\Pi^*_{I, \hat{s}_t}(\hat{s}_t, \hat{s}_t, p_t) = \left( \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial a_I} \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial \hat{s}_t} + \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial \hat{s}_t} \right) (v(\hat{s}_t) - a_I(\hat{s}_t, p_t))
- F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial \hat{s}_t}.
\]  

The first-order condition for bidder \( I \), (95), computed at \( s \downarrow \hat{s}_t \), reduces to

\[
\frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial a_I} (v(\hat{s}_t) - a_I(\hat{s}_t, p_t)) = F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t).
\]  

Plugging (106) into (105), we obtain

\[
\Pi^*_{I, \hat{s}_t}(\hat{s}_t, \hat{s}_t, p_t) = \left( \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial a_I} \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial \hat{s}_t} + \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial \hat{s}_t} \right) (v(\hat{s}_t) - a_I(\hat{s}_t, p_t)).
\]  

From (97), \( F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) = \frac{v^{-1}(a(\hat{s}_t, p_t))}{\hat{s}_t} \left( \frac{v(\hat{s}_t) - a(\hat{s}_t, p_t)}{v(\hat{s}_t) - \hat{a}(\hat{s}_t, p_t)} \right) \) immediately follows. Second, take a partial derivative of \( \Pi^*_I(s, \hat{s}_t, p_t) \) with respect to \( p_t \) and compute it at point \( s \downarrow \hat{s}_t \), using that for bidder \( I \), \( \lim_{s \downarrow \hat{s}_t} a_I(s, \hat{s}_t, p_t) = \hat{a}(\hat{s}_t, p_t) \):

\[
\Pi^*_{I, p_t}(\hat{s}_t, \hat{s}_t, p_t) = \left( \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial a_I} \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial \hat{s}_t} + \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial \hat{s}_t} \right) (v(\hat{s}_t) - a_I(\hat{s}_t, p_t))
- F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) \frac{\partial \hat{a}(\hat{s}_t, p_t)}{\partial \hat{s}_t}.
\]  

Again, using the first-order condition for bidder \( I \) at \( s \downarrow \hat{s}(t) \),

\[
\Pi^*_{I, p_t}(\hat{s}_t, \hat{s}_t, p_t) = \frac{\partial F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t)}{\partial p_t} (v(\hat{s}_t) - a_I(\hat{s}_t, p_t)).
\]  

Note that \( F_N(\hat{a}(\hat{s}_t, p_t), \hat{s}_t, p_t) \) does not depend on \( p_t \) other than through its first argument. Hence,

\[
\Pi^*_{I, p_t}(\hat{s}_t, \hat{s}_t, p_t) = 0.
\]  

Finally, by definition of \( p_t, p_t \) and \( s_t \) are linked through differential equation \( \hat{s}'_t = -\frac{p_t}{1-p_t} \lambda s_t (1 - \hat{s}_t) \).

This concludes the proof.

Equation (102) is a differential equation on \( \hat{s}_t \): \( p_t \) is a function of \( \hat{s}_t \). The initial condition is \( s_0 = 1 \). Its solution gives the evolution of \( \hat{s}_t \) over time. This equation is intuitive. For the indifferent type \( \hat{s} \), the cost of waiting equals the benefit. The cost of waiting (the left-hand side) is due to the delay in the profit from the bidder-initiated auction, the relative loss from a potential rival-initiated auction, and the decrease in the continuation value due to the decrease in cut-off \( s \) over time, which diminishes the advantage of the bidder-initiated auction. The benefit of waiting
(the right-hand side) is from the option value of experiencing a better shock.

For \( \hat{s}_t \) to be the equilibrium cut-off, we need to verify that the seller does not benefit from early initiation. At each time \( t \), the expected payoff to the seller from strategy \( \mu = 0 \), denoted by \( V_S(t,0) \), needs to satisfy

\[
rV_S(t,0) = V_{S,t}''(t,0) + 2(\lambda(1-\hat{s}_t) + p_t)R_B(\hat{s}_t, p_t),
\]

where \( R_B(\hat{s}_t, p_t) \) is the seller’s expected revenue in the bidder-initiated auction. The seller does not benefit from early initiation if and only if \( V_S(t,0) \geq R_S(\hat{s}_t) \) for any \( t \), where \( R_S(\hat{s}_t) \) is his expected revenue in the seller-initiated auction.

The equilibrium is found numerically. An additional simplification, that allows for a quasi-closed form solution without diminishing the economic insight, is to assume that when an existing bidder experiences a shock, she exits the game, obtains an exogenous exit payoff \( X > 0 \), and is replaced by a new bidder. Example 7 derived below provides such solution for the case \( v(s_t) = s_t \).

Figure 5 illustrates the behavior of the inference about the cause of initiation \( p_t \) as a function of the current cut-off \( \hat{s}_t \) and the speed of convergence to the stationary cut-off \( \hat{s} = 0.372 \) for the case \( r = 0.05, \lambda = 0.5 \), and \( X = 0.139 \) (equal to endogenous \( X(\hat{s}) \) in the similar stationary equilibrium without seller initiation, \((\hat{s}, \mu) = (0.635, 0)\), as shown in Example 3).

**Example 6: auction stage, the private-value framework, the non-stationary setting.** Suppose that \( v(s_t) = s_t \). From (100), \( \bar{a}(\hat{s}, p) = \frac{\hat{s}}{2} \). Multiplying (92) by \( \frac{1-\hat{s}}{1-p} \), multiplying (95) by \( \hat{s} \) and adding the equations up, we obtain

\[
\frac{1-\hat{s}}{1-p} F_{I,b}(\hat{s}, \hat{s}, p) (F_N(b, \hat{s}, \hat{s}) \hat{s} - b) + \hat{s} F'_{N,b}(\hat{s}, \hat{s}, p) \left( \hat{s} + \frac{F_I(b, \hat{s}, \hat{s}) - p}{1-p} (1 - \hat{s}) - b \right) = \frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p) + \hat{s} F_N(b, \hat{s}, p)
\]

\[
= d \left( \hat{s} F_N(b, \hat{s}, p) \left( \hat{s} + \frac{F_I(b, \hat{s}, \hat{s}) - p}{1-p} (1 - \hat{s}) \right) \right) = d \left( \frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p)b + \hat{s} F_N(b, \hat{s}, p)b \right)
\]

\[
\Rightarrow \hat{s} F_N(b, \hat{s}, p) \left( \hat{s} + \frac{F_I(b, \hat{s}, \hat{s}) - p}{1-p} (1 - \hat{s}) \right) = \frac{1-\hat{s}}{1-p} F_I(b, \hat{s}, p)b + \hat{s} F_N(b, \hat{s}, p)b + c,
\]

(112)

for some constant \( c \). This equation holds for \( b \in [\bar{a}(\hat{s}, p), \bar{a}(\hat{s}, p)] \). Evaluating at \( \bar{a}(\hat{s}, p) \) gives \( c \):

\[
c = \hat{s} \left( 1 - \bar{a}(\hat{s}, p) \right) - \frac{1 - \hat{s}}{1-p} \bar{a}(\hat{s}, p).
\]

(113)

Evaluating (112) at \( \hat{a}(\hat{s}, p) \) gives an alternative expression for \( c \), which we use to obtain \( \hat{a}(\hat{s}, p) \) as a function of given \( \hat{a}(\hat{s}, p) \):

\[
c = \frac{\hat{s}^2}{4} - \frac{1 - \hat{s}}{1-p} \hat{a}(\hat{s}, p) \Rightarrow \bar{a}(\hat{s}, p) = \frac{\hat{s} - \frac{\hat{s}^2}{4} + \frac{1 - \hat{s}}{1-p} \hat{a}(\hat{s}, p)}{\hat{s} + \frac{1 - \hat{s}}{1-p}}.
\]

(114)

In the stationary setting, \( p = 0 \) so \( \bar{a}(\hat{s}, p) = \frac{\hat{s}^2}{4} - \hat{s} \), as before. What remains is to solve for \( \hat{a}(\hat{s}, p) \). Plugging \( c \) into (112),

\[
\hat{s} F_N(b, \hat{s}, p) \left( \hat{s} - b + \frac{F_I(b, \hat{s}, \hat{s}) - p}{1-p} (1 - \hat{s}) \right) = \frac{1 - \hat{s}}{1-p} F_I(b, \hat{s}, p)b + \hat{s}^2 \left( 1 - \frac{1 - \hat{s}}{1-p} \hat{a}(\hat{s}, p) \right),
\]

(115)
implying that in the range \( b \in [\hat{a}(\hat{s}, p), \overline{a}(\hat{s}, p)] \),

\[
\hat{s}F_N(b, \hat{s}, p) = \frac{\frac{1-\hat{s}}{p} F_I(b, \hat{s}, p) b + \hat{s}^2 \frac{3}{4} - \frac{1-\hat{s}}{1-p} \hat{p}_a(\hat{s}, p)}{\hat{s} - b + \frac{F_I(b, \hat{s}, p)p}{1-p} (1 - \hat{s})}.
\] (116)

Inserting (116) into \( v(\cdot) \) in (92) and transforming,

\[
F'_{I,b}(b, \hat{s}, p) \left( \frac{\frac{1-\hat{s}}{p} F_I(b, \hat{s}, p) b + \hat{s}^2 \frac{3}{4} - \frac{1-\hat{s}}{1-p} \hat{p}_a(\hat{s}, p)}{\hat{s} - b + \frac{F_I(b, \hat{s}, p)p}{1-p} (1 - \hat{s})} - b \right) = F_I(b, \hat{s}, p)
\]

\[
\Rightarrow F'_{I,b}(b, \hat{s}, p) = \left( \frac{\hat{s} - b + \frac{F_I(b, \hat{s}, p)p}{1-p} (1 - \hat{s})}{\frac{3}{4} - \frac{1-\hat{s}}{1-p} \hat{p}_a(\hat{s}, p) - b\hat{s} + \hat{s}^2 + \frac{p}{1-p} (1 - \hat{s}) b} \right) F_I(b, \hat{s}, p).
\] (117)

The boundary condition is

\[
F_I(\overline{a}(\hat{s}, p), \hat{s}, p) = 1.
\] (118)

This differential equation has a closed-form solution which is omitted for brevity and available from the authors upon request. The remaining value, \( \hat{a}(\hat{s}, p) \), is numerically determined from the second boundary condition:

\[
F_I(\hat{a}(\hat{s}, p), \hat{s}, p) = p.
\] (119)

It can be shown that \( \frac{\hat{s}}{2} \leq \hat{a}(\hat{s}, p) \leq \frac{3\hat{s}}{4} \) and \( \hat{a}(\hat{s}, p) \) is strictly increasing in \( p \). In particular, \( \hat{a}(\hat{s}, 0) = \frac{\hat{s}}{2} \), so that the solution converges to that of the base model; \( \hat{a}(\hat{s}, 1) = \frac{3\hat{s}}{4} \). In addition, \( \frac{3\hat{s}}{4} \leq \overline{a}(\hat{s}, p) \) and \( \overline{a}(\hat{s}, p) \) is strictly decreasing in \( p \). In particular, \( \overline{a}(\hat{s}, 1) = \frac{3\hat{s}}{4} \), so that at \( p = 1 \), bidder \( I \) always randomizes between bids. This is because at \( p = 1, \hat{s} = 1 \), so bidder \( I \) always has exactly the cut-off signal.

**Example 7: auction stage, the private-value framework, the non-stationary setting.**

Suppose that \( v(s_t) = s_t \) and \( X(t) \equiv X \) is exogenously given. The payoffs of bidders \( I \) and \( N \) with cut-off signals are \( \Pi^*_I(\hat{s}, \hat{s}, p) = \overline{a}(\hat{s}, p) (\hat{s} - a(\hat{s}, p)) = \frac{\hat{s}}{2} \) and \( \Pi^*_N(\hat{s}, \hat{s}, p) = a(\hat{s}, \hat{s}, p) < \Pi^*_I(\hat{s}, \hat{s}, p) \).

The indifference condition for \( \hat{s}_t \), (102), simplifies to

\[
\frac{r \hat{s}_t}{4} + \frac{1 - \hat{s}_t}{1 - p_t} \left( \overline{a}(\hat{s}_t, p_t) - \frac{3\hat{s}_t}{4} \right) - \frac{p_t}{1 - p_t} \lambda \hat{s}_t (1 - \hat{s}_t) \frac{1}{2} \left( \frac{\hat{a}(\hat{s}_t, p_t) - \frac{\hat{s}_t}{2}}{\hat{s}_t - \overline{a}(\hat{s}_t, p_t)} + \frac{1}{2} \right) = \lambda \left( X - \frac{\hat{s}_t}{4} \right),
\] (120)

where \( \overline{a}(\hat{s}_t, p_t) \) and \( \hat{a}(\hat{s}_t, p_t) \) are numerically determined from (118) and (119). Equation (120) sets up a differential equation for \( \hat{s}_t \). Specifically, first, (120) is used to obtain \( p(\hat{s}_t) \); second, differential equation \( \hat{s}'_t = -\frac{p(\hat{s}_t)}{1 - p(\hat{s}_t)} \lambda \hat{s}_t (1 - \hat{s}_t) \) with the initial condition \( \hat{s}_0 = 1 \) is numerically solved.
Figure 1: Equilibrium bids and expected payoffs of bidders in a bidder-initiated common-value auction. The left panel plots the equilibrium bids as functions of signals for the initiating bidder (the normal line) and the non-initiating bidder (the dashed line). The right panel plots the corresponding expected surpluses of each bidder.

Figure 2: Equilibrium bids and expected payoffs of bidders in a bidder-initiated private-value auction. The left panel plots the equilibrium bids as functions of signals for the initiating bidder (the normal line) and the non-initiating bidder (the dashed line). The right panel plots the corresponding expected surpluses of each bidder.
Figure 3: **Equilibria with seller- and bidder-initiated auctions.** The figure plots best responses of the seller (the normal line) and each bidder (the dashed line), as well as multiple equilibrium ($\hat{s}, \mu$).

Figure 4: **Equilibrium bids in a bidder-initiated private-value auction, the non-stationary setting.** The figure plots the equilibrium bids as functions of signals for the initiating bidder (the normal line) and the non-initiating bidder (the dashed line).
Figure 5: **The dynamics** $p_t$ and $\hat{s}$. The left panel plots the behavior of the conditional probability that bidder $I$’s signal is exactly at the cut-off, $p_t$, as a function of the cut-off, $\hat{s}_t$. The right panel plots the behavior of the cut-off as a function of time. The stationary cut-off, $\hat{s}$, is 0.372.